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**A CONTROLLABLE SPACE MOTION ALONG
A BEZIER CURVE**

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Abstract

In this study, a controllable space motion is studied in $\mathbb{R}^3$ as boomerang motion. In this modeling, closed Bezier curves with n -controlled points, which the initial and the final points are same, are used. An optimal orbit can be obtained with an appropriate choosing of control points. This model motion is exemplified using Matlab.

Keywords: Bezier curve, Boomerang, Closed space curve

2010 Mathematics Subject Classifications: 53A04, 53A17, 57R25

1 Introduction

A displacement in n -dimensional space is described by $D = (A, T_d)$, where A is an orthogonal matrix and T_d is a translation with translation vector d . $D(t)$ is defined along a curve C , where $C : I \subset \mathbb{R} \rightarrow \mathbb{R}^n$. In this case, $D(t)$ is called continuous displacement and the position of a point P at time t is shown by $D(t)(P)$. The $D(t)(P)$ is called orbit of P under the displacement D .

An orbit of a point under a continuous displacement is generally a space curve [6, 11]. The orbit of a point under a displacement gives us important knowledge about the displacement. Conversely, a displacement can be defined using a space curve. In this defining properties of a curve can be used. For example, we can use a special frame as Frenet frame or Darboux frame to define a space motion [6, 9, 13, 14]. A closed space motion is a motion which every point under this motion has a closed orbit. Such a motion can be modeled as boomerang motion. If we use a closed curve then we can define a displacement along this curve. This motion looks like as boomerang because a boomerang makes a closed movement. In addition if we want to control this motion, one of the best choosing can be using a closed Bezier curve.

The boomerang motion plane form was studied [1]. When it is thrown check, it returns back to beginning point [15]. Aerodynamic performance of the boomerangs is studied [8, 12]. The boomerang must be thrown a perpendicular route for orbiting a close elliptical orbit [12]. Trajectories of the boomerang obtained in numerical experiments are very similar to the trajectories observed in reality [10].

Our goal is to answer the question: How we can define a displacement which we require? One of the best ways is using the closed Bezier curve. We define a displacement along a closed Bezier curve with n -control points in three dimensions space. A rigid body under this motion makes as a boomerang motion. The orbit is a closed Bezier curve. In this paper, we find the optimal orbit.

1.1 Bezier Curves

We know that binomial expansion with n -degree for a and b is

$$(a + b)^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i}.$$

If we take $a = u$ and $b = 1 - u$, then we can write the polynomial which is called Bernstein polynomial

$$B(u) = \sum_{i=0}^n \binom{n}{i} u^i (1 - u)^{n-i}.$$

or

$$B_n(u) = \sum_{i=0}^n B_i^n \tag{1}$$

where $B_i^n = \binom{n}{i} u^i (1-u)^{n-i}$. This polynomial was introduced by Bernstein [3]. A Bernstein polynomial can be written in matrix form [2]. So we can write $B_n(u)$

$$B_n(u) = [B_i^n]_{(n+1) \times 1} [1]_{1 \times (n+1)} \quad (2)$$

where,

$$[B_i^n] = [B_0^n \ B_1^n \ B_2^n \ \dots \ B_n^n]$$

and

$$[1]_{1 \times (n+1)} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{1 \times (n+1)}$$

Bernstein polynomials have the linearly independent property, such that, if we choose $s = \frac{u}{1-u}$ then

$$\begin{aligned} B(u) &= \sum_{i=0}^n b_i u^i (1-u)^{n-i} \\ B(u) &= \sum_{i=0}^n b_i \frac{u^i}{(1-u)^i} (1-u)^n \\ B(u) &= \sum_{i=0}^n b_i s_i \lambda \\ B(u) &= 0 \implies \forall s_i = 0. \end{aligned}$$

So B_i^n Bernstein polynomials form a basis for all polynomials of degree k , $k \leq n$ and every polynomial curve of degree k , has a unique n th degree Bezier representation

$$B_x(u) = \sum_{i=0}^n P_i B_i^n(u), \quad 0 \leq u \leq 1$$

where coefficients P_i are elements of $\mathbb{R}^k$ and are called Bezier points [4, 5, 7].

The points P_i are the vertices of the Bezier polygon of $B_x(u)$ over $[a, b]$. Any affine parameter transformation $u = a(1-t) + bt$ leaves the degree of the curve ($0 \leq t \leq 1$). The parameter t is called the local parameter and u the global parameter of $B_x(u)$.

If $P_i = (P_{ix}, P_{iy}, P_{iz}) \in \mathbb{R}^3$, $0 \leq i \leq n$, then we have

$$B_x(u) = \sum_{i=0}^n B_i^n(u) (P_{ix}, P_{iy}, P_{iz}).$$

The Bezier curve

$$B_x(u) = \left(\sum_{i=0}^n B_i^n(u) P_{ix}, \sum_{i=0}^n B_i^n(u) P_{iy}, \sum_{i=0}^n B_i^n(u) P_{iz} \right)$$

are continuous on $[a, b]$ and differentiable on (a, b) .

Using the matrix form, we can write the Bezier curve with $(n + 1)$ -point as

$$B_z(n, u) = B_n(u) \cdot \mathbf{P} \quad (3)$$

where,

$$\mathbf{P} = [P_i]_{1 \times (n+1)}$$

and $P_i = (P_{ix}, P_{iy}, P_{iz})$ are control points of $B_z(n, u)$ Bezier curve in $\mathbb{R}^3$.

1.2 Frenet Vectors of a Bezier Curve

Let $B_z(u) = \sum_{i=0}^n B_i^n(u) P_i$ be given. The derivatives of $B_z(u)$ are

$$\begin{aligned} \frac{d}{dt} B_i^n(u) &= n(B_{i-1}^{n-1}(u) - B_i^{n-1}(u)) \\ \frac{d}{du} B_z(u) &= n \sum_{i=0}^n (B_{i-1}^{n-1}(u) - B_i^{n-1}(u)) P_i \\ &= n \sum_{i=0}^{n-1} B_i^{n-1}(u) \cdot Q_i, \quad Q_i = P_{i+1} - P_i \\ \frac{d^2}{du^2} B_z(u) &= n(n-1) \sum_{i=0}^{n-2} B_i^{n-2}(u) \cdot R_i, \quad R_i = Q_{i+1} - Q_i. \end{aligned}$$

In matrix notation, if we write

$$\begin{aligned} M_2 &= [B_i^{n-1}]_{1 \times (n-1)}, \quad 0 \leq i \leq n-2 \\ N_2 &= [P_{i+2} - 2P_{i+1} + P_i]_{3 \times n}, \\ M_1 &= [B_i^{n-1}]_{1 \times (n-1)}, \quad 0 \leq i \leq n-1 \\ N_1 &= [P_{i+1} - P_i]_{3 \times n}, \end{aligned}$$

then

$$\frac{d}{du} B_z(u) = n M_1^T N_1$$

and

$$\frac{d^2}{du^2} B_z(u) = n(n-1) M_2^T N_2.$$

Suppose that $B_z(u) = \sum_{i=0}^n B_i^n(u) P_i$ be a Bezier curve with control points P_i , $0 \leq i \leq n$. Then we have the unit tangent, unit normal and unit binormal vector fields as follows

$$T(u) = \frac{B'_z(u)}{\|B'_z(u)\|},$$

$$\|B'_z(u)\| = (n^2 \sum_{i=0}^{n-1} B_i^{n-1}(u) \cdot Q_i \sum_{j=0}^{n-1} B_j^{n-1}(u) \cdot Q_j)^{\frac{1}{2}},$$

where $Q_j = P_{j+1} - P_j$.

$$\frac{d^2 B_z(u)}{du^2} = n(n-1) \sum_{i=0}^{n-2} B_i^{n-2}(u) \cdot R_i,$$

where $R_i = Q_{i+1} - Q_i$.

$$\overline{M_1} = n [B_i^{n-1}(P_{i+1} - P_i)]_{1 \times n}, \quad i = 0, 1, \dots, n-1$$

$$\overline{M_2} = (n-1) [B_j^{n-2}(P_{j+2} - 2P_{j+1} + P_j)]_{1 \times (n-1)}, \quad j = 0, 1, \dots, n-2$$

M_1 and M_2 are two matrices which all entries are vectors of dimension 3.

Definition 1. Let A and B be two matrices with all entries are vectors of the dimension of $\mathfrak{B}$, and A and B be productive. Then the product

$$A \times B = [ab_{ij}]$$

where $ab_{ij} = a_i \times b_j$ is vector product in $\mathbb{R}^3$, is called the vectorial product of A and B matrices.

Consequently we have,

$$\begin{aligned} B'_z(t) \times B''_z(t) &= [1]_{1 \times n}^t (\overline{M_1} \times \overline{M_2}) \times [1]_{1 \times (n-1)} \\ &= n(n-1) [B_i^{n-1} B_j^{n-2} (P_{i+1} - P_i) \times (P_{j+2} - 2P_{j+1} + P_j)]. \end{aligned}$$

$$B = B'_z(u) \times B''_z(u) \frac{1}{\|B'_z(u) \times B''_z(u)\|}$$

and

$$N(u) = B(u) \times T(u).$$

$$B = \frac{\frac{d}{du} B_z(n, u) \times \frac{d^2}{du^2} B_z(n, u)}{\left\| \frac{d}{du} B_z(n, u) \times \frac{d^2}{du^2} B_z(n, u) \right\|}$$

and

$$N = B \times T$$

For $B(u) = (b_1(u), b_2(u), b_3(u))$, using the Cayley's formula, we have

$$R = \begin{bmatrix} b_1(u)^2(1 - c\theta) + c\theta & b_1(u)b_2(u)(1 - c\theta) - b_3(u)s\theta & b_1(u)b_3(u)(1 - c\theta) + b_2(u)s\theta \\ b_1(u)b_2(u)(1 - c\theta) + b_3(u)s\theta & b_2(u)^2(1 - c\theta) + c\theta & b_2(u)b_3(u)(1 - c\theta) - b_1(u)s\theta \\ b_1(u)b_3(u)(1 - c\theta) - b_2(u)s\theta & b_2(u)b_3(u)(1 - c\theta) + b_1(u)s\theta & b_3(u)^2(1 - c\theta) + c\theta \end{bmatrix}$$

is a rotation matrix with rotating axis is $B(u)$, for all u , where $c\theta = \cos \theta$, $s\theta = \sin \theta$.

In addition, for every point of $B_z(u)$, if we take the point $B_z(u)$ is a pole point then

$$D = (R, (I - R)B_z(u))$$

is a displacement along the $B_z(u)$ curve.

2 Model Motion as a Boomerang in Space

In this section, we introduce a model motion, like boomerang motion. When we defined this motion, we inspired from boomerang act. In this motion, the orbit is a Bezier curve with n -control points, the rotating axis is binormal vector fields of the Bezier curve, and a boomerang deploys on tangent line (or normal line) at the beginning.

Modeling the boomerang motion along the $B_*(u)$ Bezier curve, we can choose the $[AB]$ segment line as the boomerang, where $A = OB_*(u) - B_*(u)N$ and $B = OB_*(u) + B_*(u)N$.

For example;

$$\begin{aligned} A &= P_0 = (a_1, a_2, a_3) = (P_{01}, P_{02}, P_{03}) \\ B &= P_1 = (b_1, b_2, b_3) = (P_{11}, P_{12}, P_{13}) \\ C &= P_2 = (c_1, c_2, c_3) = (P_{21}, P_{22}, P_{23}) \\ D &= P_3 = (d_1, d_2, d_3) = (P_{31}, P_{32}, P_{33}) \\ E &= P_4 = (e_1, e_2, e_3) = (P_{41}, P_{42}, P_{43}) \\ F &= P_5 = (f_1, f_2, f_3) = (P_{51}, P_{52}, P_{53}) \end{aligned}$$

For $n = 5$,

$$\begin{aligned} B_*(5, t) &= \sum_{i=0}^5 \binom{5}{i} t^i (1-t)^{5-i} P_i \\ &= (1-t)^5 P_0 + t(1-t)^4 P_1 + t^2(1-t)^3 P_2 \\ &\quad + t^3(1-t)^2 P_3 + t^4(1-t) P_4 + t^5 P_5 \\ &= (1-t)^5 (a_1, a_2, a_3) + 5t(1-t)^4 (b_1, b_2, b_3) + 10t^2(1-t)^3 (c_1, c_2, c_3) \\ &\quad + 10t^3(1-t)^2 (d_1, d_2, d_3) + 5t^4(1-t) (e_1, e_2, e_3) + t^5 (f_1, f_2, f_3) \end{aligned}$$

$$B_*(t) = \begin{bmatrix} (1-t)^5 & 5t(1-t)^4 & 10t^2(1-t)^3 & 10t^3(1-t)^2 & 5t^4(1-t) & t^5 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \end{bmatrix}.$$

Now we will calculate the derivative of $B_*(t)$

$$B'_*(t) = n \sum_{i=0}^{n-1} \binom{n-1}{i} t^i (1-t)^{(n-1)-i} (P_{i+1} - P_i).$$

So we have

$$B'_*(t) = 5 \begin{bmatrix} (1-t)^4 & 4t(1-t)^3 & 6t^2(1-t)^2 & 4t^3(1-t) & t^4 \end{bmatrix} \begin{bmatrix} b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ c_1 - b_1 & c_2 - b_2 & c_3 - b_3 \\ d_1 - c_1 & d_2 - c_2 & d_3 - c_3 \\ e_1 - d_1 & e_2 - d_2 & e_3 - d_3 \\ f_1 - e_1 & f_2 - e_2 & f_3 - e_3 \end{bmatrix}.$$

Finally, we will calculate $B'_z(t)$ according to

$$B''_z(t) = n(n-1) \sum_{i=0}^n B_i^{n-2} (P_{i+2} - 2P_{i+1} + P_i).$$

So we have

$$B''_z(t) = 5.4. \begin{bmatrix} (1-t)^3 & 3t(1-t)^2 & 3t^2(1-t) & t^3 \end{bmatrix} \cdot \begin{bmatrix} c_1 - 2b_1 + a_1 & c_2 - 2b_2 + a_2 & c_3 - 2b_3 + a_3 \\ d_1 - 2c_1 + b_1 & d_2 - 2c_2 + b_2 & d_3 - 2c_3 + b_3 \\ e_1 - 2d_1 + c_1 & e_2 - 2d_2 + c_2 & e_3 - 2d_3 + c_3 \\ f_1 - 2e_1 + d_1 & f_2 - 2e_2 + d_2 & f_3 - 2e_3 + d_3 \end{bmatrix}$$

2.1 Matlab Programme According to Boomerang Motion Along Bezier Curve

m.file according to this motion is as follows. The boomerang motion along the Bezier curve is drawn as shown in Figure 1.

```
close all, clear all, clc
a=50;
axis([-5 a -5 a -5 a])
axis equal
axis square
hold on
xlabel('x axis');
ylabel('y axis');
zlabel('z axis');
hold on
figure(1)
x_log=[];
y_log=[];
z_log=[];
for t=0:0.05:1
a1=20; %A, P0
a2=15;
a3=7;
b1=50; %B, P1
b2=40;
b3=20
c1=30;%C P2
c2=45;
c3=39;
d1=13;%D, P3
```

```

d2=44;
d3=15;
e1=2;%E, P4
e2=17;
e3=19;
f1=20;%F,P5
f2=15;
f3=7;
A=[ a1 a2 a3] % A is P0;
text(a1,a2,a3,'P0')
B=[b1 b2 b3] % B is P1;
C=[c1 c2 c3] % C is P2;
D=[d1 d2 d3] % D is P3;
E=[e1 e2 e3] % E is P4;
F=[f1 f2 f3] % F is P5
text(a1,a2,a3,'P0');
text(b1,b2,b3,'P1');
text(c1,c2,c3,'P2');
text(d1,d2,d3,'P3');
text(e1,e2,e3,'P4');
text(f1,f2,f3,'P5');
Bz=[(1-t)^5 5*t*(1-t)^4 10*t^2*(1-t)^3 10*t^3*(1-t)^2 5*t^4*(1-t) t^5];
P=[A;B;C;D;E;F];
Crv=Bz*P % Bezier curve
Cr=[Crv(1); Crv(2);Crv(3)]
x=Crv(1);
y=Crv(2);
z=Crv(3);
plot3(x_log,y_log,z_log,'r')
PP=[B-A;C-B;D-C;E-D;F-D] %PP is Q
dBz=5*[(1-t)^4 4*t*(1-t)^3 6*t^2*(1-t)^2 4*t^3*(1-t) t^4]
dBz*PP;
ddBz=[(1-t)^3 3*t*(1-t)^2 3*t^2*(1-t) t^3]
PPP=[C-2*B+A;D-2*C+B;E-2*D+C;F-2*E+D]
hold on
figure(1);
plot3(Crv(1), Crv(2), Crv(3), 'r');
pause(0.01)
T=(dBz*PP)/norm(dBz*PP); %tangent
BB=cross(dBz*PP,ddBz*PPP)/norm(cross(dBz*PP,ddBz*PPP));
N=cross(BB,T);
a=BB(1);
b=BB(2);
c=BB(3);
norm(N);

```

```
hold on
for Q=1:120:720
I=[1 0 0;0 1 0;0 0 1];
r=[a*a*(1-cosd(Q))+cosd(Q) a*b*(1-cosd(Q))-c*sind(Q) a*c*(1-cosd(Q))+b*sind(Q)
a*b*(1-cosd(Q))+c*sind(Q) b*b*(1-cosd(Q))+cosd(Q) b*c*(1-cosd(Q))-a*sind(Q)
a*c*(1-cosd(Q))-b*sind(Q) b*c*(1-cosd(Q))+a*sind(Q) c*c*(1-cosd(Q))+cosd(Q)];
pol=(I-r)*Cr; %pole point of motion
R=[a*a*(1-cosd(Q))+cosd(Q) a*b*(1-cosd(Q))-c*sind(Q) a*c*(1-cosd(Q))+b*sind(Q)
a*b*(1-cosd(Q))+c*sind(Q) b*b*(1-cosd(Q))+cosd(Q) b*c*(1-cosd(Q))-a*sind(Q)
a*c*(1-cosd(Q))-b*sind(Q) b*c*(1-cosd(Q))+a*sind(Q) c*c*(1-cosd(Q))+cosd(Q)
0 0 0 1];
x_log=[x_log x];
y_log=[y_log y];
z_log=[z_log z];
AAa=[Crv(1);Crv(2);Crv(3);1]-[N(1);N(2);N(3);1]
norm([Crv(1);Crv(2);Crv(3);1]-[N(1);N(2);N(3);1])
BBb=[Crv(1);Crv(2);Crv(3);1]+[N(1);N(2);N(3);1]
norm([Crv(1);Crv(2);Crv(3);1]+[N(1);N(2);N(3);1])
Aa=R*AAa
Bb=R*BBb
line([Aa(1),Bb(1)],[Aa(2),Bb(2)],
[Aa(3),Bb(3)], 'Marker', 'r', 'LineStyle', '-',
'Color', 'r', 'LineWidth', 1)
line([A(1),B(1)],[A(2),B(2)], [A(3),B(3)])
line([C(1),B(1)],[C(2),B(2)], [C(3),B(3)])
line([C(1),D(1)],[C(2),D(2)], [C(3),D(3)])
line([E(1),D(1)],[E(2),D(2)], [E(3),D(3)])
line([E(1),F(1)],[E(2),F(2)], [E(3),F(3)])
pause(0.1)
end
end
```

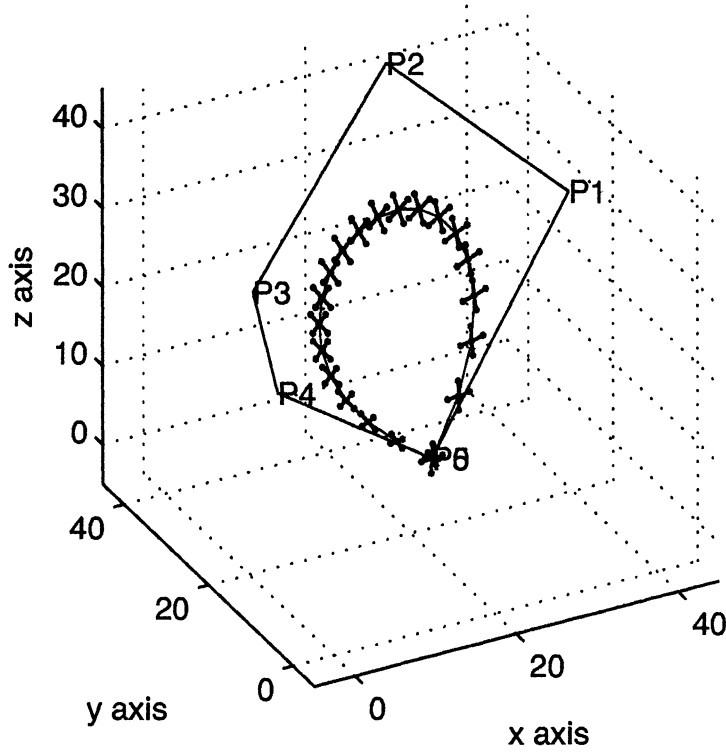


Figure 1: The boomerang motion along the Bezier curve

3 Conclusions

Choosing the curve for a displacement along a curve is very important. Some spatial displacement in kinematics and mechanics have a lot of special properties. If we choose the curve in the space as a Bezier curve with n -control points, then we can define a displacement as we desired. So using this facts, we define a new motion in $\mathbb{R}^3$ as a boomerang. The orbit of the motion we defined is a Bezier curve. We use the unit binormal vector of the Bezier curve as a rotating axis of rotation. A line segment located on the unit normal vector acts as a boomerang. This paper introduces a different and new approach to the closed space motion. When we use a closed Bezier curve, the orbit of the boomerang is optimal orbit for an original boomerang. In addition, we illustrate this motion in Matlab.

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PRESENTATION OF LINEAR GROUPS
 $GL(2, \mathbb{Z}_n)$

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Abstract

In this article, we have given Lie regular generators of linear group $GL(2, \mathbb{Z}_n)$. Using these generators we have established a presentation of linear group $GL(2, \mathbb{Z}_n)$ for $n \in \mathbb{N}$, $n > 1$.

Keywords: Lie regular units; Linear group; Presentation of a group.

1 Introduction

Let $GL(n, R)$ denotes the general linear group of $n \times n$ invertible matrices and $SL(n, R)$ denotes the special linear group of $n \times n$ matrices with determinant 1 over a ring R . Finding a generating set and a presentation of a group is always considered to be a nice problem. Here we recall some work done related to generators and presentation of $GL(n, R)$. The linear group $GL(n, \mathbb{Z})$ is called the unimodular group and the set of generators has been found by C. C. Macduffee (See [8], p. 34). Trott has reduced the number of generators for unimodular group (see [10]). When $R = \mathbb{F}_p$ is a finite field with characteristic p , a presentation of $GL(2, R)$ is available in [2, p. 95-96]. In 1994, T. A. Francis has established presentations of $SL(n, R)$ and $GL(n, R)$, where R is a division ring (see [3]). Pooja et al. introduced Lie regular elements and Lie regular units over a ring R (see [9]). They have given generators and presentations for linear groups $GL(2, \mathbb{Z}_n)$ for some positive integers n (See [6, 9]). In this article we consider $GL(2, R)$, where R is a ring of integers modulo n , which is not a division ring. We show that these Lie regular elements generate $GL(2, \mathbb{Z}_n) \forall n > 1, n \in \mathbb{N}$ (See section 3). In section 4, we give a presentation of $GL(2, \mathbb{Z}_{2^n})$ (See Theorem 4.2) and $GL(2, \mathbb{Z}_{p^n})$ (See Theorem 4.3) with lesser number of generators and relations than those previously known.

Throughout this paper, ϕ denotes the Euler's totient function and $\mathcal{U}(R)$ denotes the unit group of the ring R . Suppose G be a group then $o(G)$ denotes the order of G .

2 Preliminaries

We summarize some well known results those will be needed subsequently and for basic results of group theory see [5].

Theorem 2.1. [4, pg. 1687] Let $t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $s = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, then

$SL(2, \mathbb{Z}_{p^r})$ has the following defining relations

$$s^{p^r} = t^4 = 1, (st)^3 = t^2, (s^\lambda ts^{\lambda^{-1}}t)^3 = 1, (s^{2\lambda}ts^{\lambda^{-1}}t)^2 = t^2,$$

where $\lambda \in \mathcal{U}(\mathbb{Z}_{p^r})$ such that,

$$o(\lambda) = \begin{cases} 2^{r-2} & \text{if } p = 2 \text{ and } r \geq 3 \\ \phi(p^r) & \text{if } p \text{ is odd or } p^r = 2 \text{ or } 4 \end{cases}.$$

Lemma 2.1. The matrices $s = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ generate $SL(2, \mathbb{Z})$.

Proof. Firstly observe that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} t = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} s^k = \begin{pmatrix} a + bk & b \\ c + dk & d \end{pmatrix}.$$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2, \mathbb{Z})$. Suppose $c \neq 0$ and $|c| \geq |d|$ write $c = dq + r$ with $0 \leq r < |d|$. Then $\gamma s^{-q} t$ has lower right entry r and $r < |d|$. We apply the above process until we get the matrix with lower left entry is zero, such a matrix has the form $\begin{pmatrix} \pm 1 & k \\ 0 & \pm 1 \end{pmatrix} = ts^{-k}t^{-1}$ or $t^{-1}s^kt^{-1}$. $\square$

Lemma 2.2. The natural map $f : SL(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z}_n)$ is onto.

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}_n)$, we see that $\gcd(a, b, n) = 1$. Assume that $a \neq 0$. Choose $t = \prod_{p|a, p \nmid b} p$, then $\gcd(a, b') = 1$, where $b' = b + tn$. There exist $x, y \in \mathbb{Z}$ such that $ax - b'y = 1$, set $c' = c + y(1 - (ad - b'c))$, $d' = d + x(1 - (ad - b'c))$. $\square$

Lemma 2.3. *Let m, n be two positive integers such that $\gcd(m, n) = 1$. Then $\mathcal{U}(\mathbb{Z}_{mn}) \cong \mathcal{U}(\mathbb{Z}_m) \times \mathcal{U}(\mathbb{Z}_n)$.*

Theorem 2.2. *Suppose p is a prime. Then order of the linear group $GL(2, \mathbb{Z}_{p^n})$ is $p^{2n-1}(p+1)(\phi(p^n))^2$.*

Corollary 2.1. *Suppose $n = \prod_{i=1}^k p_i^{r_i}$, where p_i 's are distinct primes, the order of $GL(2, \mathbb{Z}_n)$ is $\prod_{i=1}^k o(GL(2, \mathbb{Z}_{p_i^{r_i}}))$.*

Theorem 2.3. *The order of the linear group $SL(2, \mathbb{Z}_n)$ is $\frac{o(GL(2, \mathbb{Z}_n))}{\phi(n)}$.*

Proof. Consider the map $f : GL(2, \mathbb{Z}_n) \rightarrow \mathcal{U}(\mathbb{Z}_n)$ such that

$$f(A) = \det(A).$$

□

Theorem 2.4. [7, Theorem 1.1] *Two groups having same presentation are isomorphic.*

Definition 2.1. [9, Definition 2.1] *An element 'a' of a ring R is said to be Lie regular if $a = [e, u] = eu - ue$, where e is an idempotent of R and u is a unit of R . Further, a unit in R is said to be Lie regular unit if it is Lie regular as an element of R .*

Proposition 2.1. *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_n)$ be a Lie regular element.*

Suppose Δ denotes the determinant of A and there exists an element $\alpha \in \mathbb{Z}_n$, such that $\alpha^2 \equiv -\Delta \pmod{n}$. Then A has an eigenvalue in $\mathbb{Z}_n$, if either b or c is a unit.

Proof. Since A is a Lie regular element, so it has the form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. We see that $\alpha, -\alpha \in \mathbb{Z}_n$ are roots of its characteristic polynomial. Suppose $v = (x, y)$ is a non-zero element in $\mathbb{Z}_n \times \mathbb{Z}_n$. By $v(\alpha I - A) = (0, 0)$, we get if b is a unit, then $x = b^{-1}(\alpha + a)y$, where $x \in \mathbb{Z}_m$. Choose $y = 1$, then a non-zero element $v = (b^{-1}(\alpha + a), 1) \in \mathbb{Z}_n \times \mathbb{Z}_n$ satisfies $vA = v\alpha$. Hence α is an eigenvalue of A . If c is a unit, similar argument holds and a non-zero vector $v = (1, c^{-1}(\alpha - a)) \in \mathbb{Z}_n \times \mathbb{Z}_n$ is an eigenvector. □

Example. $A = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \in M_2(\mathbb{Z}_n)$, here $\Delta = -1$, $\alpha = 1$ or -1 , then corresponding eigenvectors are $v = (1, 0)$ and $v = (1, 2)$ respectively.

Observe that the element

$$a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} = \left[\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \right]$$

is a Lie regular unit in $M_2(\mathbb{Z}_n)$ and $b = \begin{pmatrix} 0 & k \\ 1 & 0 \end{pmatrix}$, where k is invertible in $\mathbb{Z}_n$, is also a Lie regular unit in $M_2(\mathbb{Z}_n)$ (see [9, Proposition 2.14]).

3 Generators of General Linear Groups

We know that a presentation of a group is one way to study and describe the structure of a group. Suppose G_1 and G_2 are two groups such that $G_1 = \langle S_{G_1} \mid R_{G_1} \rangle$ and $G_2 = \langle S_{G_2} \mid R_{G_2} \rangle$, where S_{G_1}, S_{G_2} are generating sets and R_{G_1}, R_{G_2} are defining relations. Then generating set of $G_1 \times G_2$ is $S_{G_1} \cup S_{G_2}$. It is well known, that if m, n are relatively positive prime integers then $GL(2, \mathbb{Z}_{mn}) \cong GL(2, \mathbb{Z}_m) \times GL(2, \mathbb{Z}_n)$. So we can find a generating set for $GL(2, \mathbb{Z}_{mn})$ but if we want to reduce the number of generators for the same then we have to do some tedious calculation. Here we are going to provide a generating set for $GL(2, \mathbb{Z}_n) \forall n > 1$. In our results the number of generators depends on the number of primes in the factorization of n .

Theorem 3.1. Suppose m is an odd positive integer such that $m = \prod_{i=1}^k p_i^{r_i}$. Let $\alpha_i \in \mathcal{U}(\mathbb{Z}_{2m})$ be such that α_i is a primitive element modulo $p_i^{r_i}$ in $\mathbb{Z}_{2m}$ respectively i.e. $o(\alpha_i) = \phi(p_i^{r_i})$ and $\prod_{i=1}^k \alpha_i^{j_i} \not\equiv 1 \pmod{2m}$, for $0 \leq j_i < \phi(p_i^{r_i})$, where $j_1, j_2 \dots j_k$ are not simultaneously 0. Then the linear group $GL(2, \mathbb{Z}_{2m})$ is generated by Lie regular units a, b, c and d_i

for $1 \leq i \leq k$, where

$$a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, d_i = \begin{pmatrix} 0 & \alpha_i \\ 1 & 0 \end{pmatrix}.$$

Proof. By Lemma 2.3, there exist elements $\alpha_i \in \mathcal{U}(\mathbb{Z}_{2m})$ such that $o(\alpha_i) = \phi(p_i^{r_i})$ and $\prod_{i=1}^k \alpha_i^{j_i} \not\equiv 1 \pmod{2m}$, for $0 \leq j_i < \phi(p_i^{r_i})$. Let G be a subgroup of $GL(2, \mathbb{Z}_{2m})$ generated by a, b, c and d_i . Set $x = bca = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Then by using Lemmas 2.1 and 2.2, x and c^{-1} generate $SL(2, \mathbb{Z}_{2m})$.

Now suppose $y_i = bd_i = \begin{pmatrix} 1 & 0 \\ 0 & \alpha_i \end{pmatrix}$. Then $y_i \in GL(2, \mathbb{Z}_{2m})$, also order of $y_i = \phi(p_i^{r_i})$. Consider

$$H = \langle y_1, y_2, \dots, y_k | y_i^{\phi(p_i^{r_i})}, y_i y_j = y_j y_i, 1 \leq i, j \leq k \rangle.$$

Then H is a subgroup of G . Since $\prod_{i=1}^k \alpha_i^{j_i} \not\equiv 1 \pmod{2m}$, clearly $\prod_{i=1}^k y_i^{j_i} \not\equiv 1 \pmod{2m}$. Thus canonical form of $H = \langle \prod_{i=1}^k y_i^{j_i} | 0 \leq j_i < \phi(p_i^{r_i}) \rangle$ and order of H is $\prod_{i=1}^k \phi(p_i^{r_i})$. We can directly see that an arbitrary element of H is of the form $\begin{pmatrix} 1 & 0 \\ 0 & \prod_{i=1}^k \alpha_i^{j_i} \end{pmatrix}$ with determinant $\prod_{i=1}^k \alpha_i^{j_i}$. Since $\prod_{i=1}^k \alpha_i^{j_i} \not\equiv 1 \pmod{2m}$, hence $H \cap SL(2, \mathbb{Z}_{2m}) = \{I_2\}$. Thus

$$o(HSL(2, \mathbb{Z}_{2m})) = o(H)o(SL(2, \mathbb{Z}_{2m}))$$

and $o(HSL(2, \mathbb{Z}_{2m})) = \prod_{i=1}^k \phi(p_i^{r_i})o(SL(2, \mathbb{Z}_{2m}))$. Since $HSL(2, \mathbb{Z}_{2m}) \subseteq G \leq GL(2, \mathbb{Z}_{2m})$ and $o(GL(2, \mathbb{Z}_{2m})) = \prod_{i=1}^k \phi(p_i^{r_i})o(SL(2, \mathbb{Z}_{2m}))$ (by Theorem 2.3), we get $HSL(2, \mathbb{Z}_{2m}) = GL(2, \mathbb{Z}_{2m})$. Hence result follows. $\square$

Theorem 3.2. Suppose m is an odd positive integer such that $m = \prod_{i=1}^k p_i^{r_i}$. Let $\alpha_i \in \mathcal{U}(\mathbb{Z}_m)$, such that α_i is a primitive element modulo $p_i^{r_i}$ in $\mathbb{Z}_m$ respectively i.e. $o(\alpha_i) = \phi(p_i^{r_i})$ and $\prod_{i=1}^k \alpha_i^{j_i} \not\equiv 1 \pmod{m}$, for $0 \leq j_i < \phi(p_i^{r_i})$, where $j_1, j_2 \dots j_k$ are not simultaneously 0. Then the linear group $GL(2, \mathbb{Z}_m)$ is generated by Lie regular units a, b, c and d_i for

$1 \leq i \leq k$, where

$$a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, d_i = \begin{pmatrix} 0 & \alpha_i \\ 1 & 0 \end{pmatrix}.$$

Proof. Proof of this theorem is similar to the above theorem.

Observe that if $m = p^k$ then generating set for $GL(2, \mathbb{Z}_m)$ is same as given in [9]. $\square$

Theorem 3.3. Suppose m is an odd positive integer such that $m = \prod_{i=1}^k p_i^{r_i}$. Let $\alpha_i \in \mathcal{U}(\mathbb{Z}_{4m})$, such that α_i is a primitive element modulo $p_i^{r_i}$ in $\mathbb{Z}_{4m}$ i.e. $o(\alpha_i) = \phi(p_i^{r_i})$ with $(\prod_{i=1}^k \alpha_i^{j_i}) \not\equiv \pm 1 \pmod{4m}$ for $0 \leq j_i < \phi(p_i^{r_i})$, where $j_1, j_2, \dots, j_k$ are not simultaneously 0. Then the linear group $GL(2, \mathbb{Z}_{4m})$ is generated by Lie regular units a, b, c , and d_i for $1 \leq i \leq k$, where

$$a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, d_i = \begin{pmatrix} 0 & \alpha_i \\ 1 & 0 \end{pmatrix}.$$

Proof. By using Lemma 2.3, there exist elements $\alpha_i \in \mathcal{U}(\mathbb{Z}_{4m})$ such that $(\prod_{i=1}^k \alpha_i^{j_i}) \not\equiv \pm 1 \pmod{4m}$ and $o(\alpha_i) = \phi(p_i^{r_i})$. Let G be a subgroup of $GL(2, \mathbb{Z}_{4m})$ generated by a, b, c and d_i . Set $x = bca = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then by using Lemmas 2.1 and 2.2, x and c^{-1} generate $SL(2, \mathbb{Z}_{4m})$. Now Suppose $y = bc = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $z_i = bd_i = \begin{pmatrix} 1 & 0 \\ 0 & \alpha_i \end{pmatrix}$. Then the order of y and z_i is 2 and $\phi(p_i^{r_i})$ respectively. Consider

$$H = \langle y, z_i \mid y^2, z_i^{\phi(p_i^{r_i})}, yz_i = z_iy, z_iz_r = z_rz_i, 1 \leq i, r \leq k \rangle.$$

Then H is a subgroup of G . Since $y \not\equiv z_i^{j_i} \pmod{4m}$. Thus the canonical form of $H = \langle y^j z_1^{j_1} \dots z_k^{j_k} \mid j = 0, 1, 0 \leq j_i < \phi(p_i^{r_i}) \rangle$ and the order of $H = 2\phi(m)$. We can directly see that any arbitrary element of H has the form $\begin{pmatrix} 1 & 0 \\ 0 & \pm(\prod_{i=1}^k \alpha_i^{j_i}) \end{pmatrix}$ with determinant $\pm(\prod_{i=1}^k \alpha_i^{j_i})$. Since

$(\prod_{i=1}^k \alpha_i^{j_i}) \not\equiv \pm 1 \pmod{4m}$, hence $H \cap SL(2, \mathbb{Z}_{4m}) = \{I_2\}$. Thus

$$o(HSL(2, \mathbb{Z}_{4m})) = o(H)o(SL(2, \mathbb{Z}_{4m}))$$

and $o(HSL(2, \mathbb{Z}_{4m})) = 2\phi(m)o(SL(2, \mathbb{Z}_{4m}))$. Since $HSL(2, \mathbb{Z}_{4m}) \subseteq G \leq GL(2, \mathbb{Z}_{4m})$ and $o(GL(2, \mathbb{Z}_{4m})) = 2\phi(m)o(SL(2, \mathbb{Z}_{4m}))$ (by Theorem 2.3), we get $HSL(2, \mathbb{Z}_{4m}) = GL(2, \mathbb{Z}_{4m})$. Hence result follows. $\square$

Proposition 3.1. *Consider the group $\mathcal{U}(\mathbb{Z}_{2^n})$. For $n > 2$, the order of $\mathcal{G}$ is 2^{n-2} .*

Proof. For $n = 3$ we can directly see that $o(3) = 2$, now for $n \geq 4$ first we will show that $3^{2^{k-3}} \equiv 1 + 2^{k-1} \pmod{2^k}$. For $k = 4$ we have $3^2 \equiv 1 + 2^3 \pmod{2^4}$. Suppose result holds for $k = m$ i.e. $3^{2^{m-3}} \equiv 1 + 2^{m-1} \pmod{2^m}$. Hence there exist an integer r so that

$$\begin{aligned} 3^{2^{m-3}} &= 1 + 2^{m-1} + r2^m \\ 3^{2^{m-2}} &= (1 + 2^{m-1} + r2^m)^2 \\ 3^{2^{m-2}} &\equiv 1 + 2^m \pmod{2^{m+1}} \not\equiv 1 \pmod{2^{m+1}} \end{aligned}$$

Hence we get $o(3) \geq 2^{m-1}$ and result follows, since $\mathcal{U}(\mathbb{Z}_{2^{m+1}})$ is not cyclic. $\square$

Theorem 3.4. *Suppose m is an odd positive integer such that $m = \prod_{i=1}^k p_i^{r_i}$ and $n > 2$. Let $\alpha, \alpha_i \in \mathcal{U}(\mathbb{Z}_{2^{nm}})$, such that $o(\alpha) = 2^{n-2}$ and α_i is a primitive element modulo $p_i^{r_i}$ in $\mathbb{Z}_{2^{nm}}$ i.e. $o(\alpha_i) = \phi(p_i^{r_i})$ with $(\prod_{i=1}^k \alpha_i^{j_i}) \alpha^j \not\equiv \pm 1 \pmod{2^{nm}}$ for $0 \leq j_i < \phi(p_i^{r_i})$ and $0 \leq j < 2^{n-2}$, where $j_1, j_2 \dots j_k, j$ are not simultaneously 0. Then the linear group $GL(2, \mathbb{Z}_{2^{nm}})$ is generated by Lie regular units a, b, c, d and e_i for $1 \leq i \leq k$, where*

$$a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, d = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} \text{ and } e_i = \begin{pmatrix} 0 & \alpha_i \\ 1 & 0 \end{pmatrix}.$$

Proof. By Lemma 2.3 and Proposition 3.1, we have elements $\alpha, \alpha_i \in \mathcal{U}(\mathbb{Z}_{2^n m})$ with $o(\alpha) = 2^{n-2}$ and $o(\alpha_i) = \phi(p_i^{r_i})$, such that $(\prod_{i=1}^k \alpha_i^{j_i}) \alpha^j \not\equiv \pm 1 \pmod{2^n m}$. Suppose G is a subgroup of $GL(2, \mathbb{Z}_{2^n m})$ generated by a, b, c, d and e_i . Set $x = bca = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then by using Lemmas 2.1 and

2.2, x and c^{-1} generate $SL(2, \mathbb{Z}_{2^n m})$. Now Suppose $y = bc = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$z = bd = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ and $w_i = be_i = \begin{pmatrix} 1 & 0 \\ 0 & \alpha_i \end{pmatrix}$. Then the order of y, z and w_i is $2, 2^{n-2}$ and $\phi(p_i^{r_i})$ respectively. Consider

$$H = \langle y, z, w_i \mid y^2, z^{2^{n-2}}, w_i^{\phi(p_i^{r_i})}, yz = zy, yw_i = w_i y, zw_i = w_i z, w_i w_r = w_r w_i, 1 \leq i, r \leq k \rangle.$$

Then H is a subgroup of G . Since $y \not\equiv z^j \not\equiv w_i^{j_i} \pmod{2^n m}$. Thus the canonical form of $H = \langle y^l z^j w_1^{j_1} \dots w_k^{j_k} \mid l = 0, 1, 0 \leq j < 2^{n-2}, 0 \leq j_i < \phi(p_i^{r_i}) \rangle$ and the order of $H = 2^{n-1} \phi(m)$. We can directly see that an arbitrary element of H has the form $\begin{pmatrix} 1 & 0 \\ 0 & \pm(\prod_{i=1}^k \alpha_i^{j_i}) \alpha^j \end{pmatrix}$ with determinant $\pm(\prod_{i=1}^k \alpha_i^{j_i}) \alpha^j$. Since $(\prod_{i=1}^k \alpha_i^{j_i}) \alpha^j \not\equiv \pm 1 \pmod{2^n m}$, hence $H \cap SL(2, \mathbb{Z}_{2^n m}) = \{I_2\}$. Thus

$$o(HSL(2, \mathbb{Z}_{2^n m})) = o(H)o(SL(2, \mathbb{Z}_{2^n m})),$$

$o(HSL(2, \mathbb{Z}_{2^n m})) = 2^{n-1} \phi(m) o(SL(2, \mathbb{Z}_{2^n m}))$. Since $HSL(2, \mathbb{Z}_{2^n m}) \subseteq G \leq GL(2, \mathbb{Z}_{2^n m})$ and $o(GL(2, \mathbb{Z}_{2^n m})) = 2^{n-1} \phi(m) o(SL(2, \mathbb{Z}_{2^n m}))$ (by using Theorem 2.3), we get $HSL(2, \mathbb{Z}_{2^n m}) = GL(2, \mathbb{Z}_{2^n m})$. Hence result follows. $\square$

Thus the above results show that Lie regular units generate $GL(2, \mathbb{Z}_n)$ $\forall n \in \mathbb{N}, n > 1$. The exciting thing is that the generators have a fixed form $\begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$ for some $\alpha \in \mathcal{U}(\mathbb{Z}_n)$, except $\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$.

4 Presentation of General linear groups

Note that $GL(2, \mathbb{Z}_2)$ is same as $SL(2, \mathbb{Z}_2)$. So presentation of $GL(2, \mathbb{Z}_2)$ is given by

$$\langle a, b \mid a^2, b^2, (ab)^3 \rangle,$$

where $a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are Lie regular elements (see Theorem 2.1).

Theorem 4.1. *Let*

$$a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then a presentation of $GL(2, \mathbb{Z}_4)$ is

$$\langle a, b, c \mid a^2, b^2, c^4, c^2 \in \text{center}, (ab)^3, (bc)^2, (cab)^4 \rangle.$$

Proof. Let G be a group generated by a, b, c and having presentation,

$$\langle a, b, c \mid a^2, b^2, c^4, c^2 \in \text{center}, (ab)^3, (bc)^2, (bca)^4 \rangle.$$

First we shall show that G is finite. Consider a subgroup H of G having a presentation is given by

$$H = \langle s, t \mid s^4, t^4, (st)^3 = t^2, (s^3ts^{3^{-1}}t)^3 = 1, (s^2ts^{3^{-1}}t)^2 = t^2 \rangle,$$

where $s = bca$ and $t = c^{-1}$. By using above relators, we can see that H is normal subgroup of G and H is isomorphic to $SL(2, \mathbb{Z}_4)$ (see Theorem 2.1 and Theorem 2.4). So H is finite. Further we have,

$$G/H \cong \langle b \mid b^2 \rangle. \quad (1)$$

Therefore G is finite. Now Suppose $x = bc$. Then by using above relators, the order of x is 2. Consider

$$K = \langle x \mid x^2 \rangle.$$

Then K is a subgroup of G and order of K is 2. We can directly see that an arbitrary element of K has the form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, hence $H \cap K = \{I_2\}$.

Thus

$$o(HK) = o(H)o(K).$$

Since $(HK) \subseteq G$ and $o(GL(2, \mathbb{Z}_4)) = 2o(SL(2, \mathbb{Z}_4))$ (by using Theorem 2.3), we get $G = GL(2, \mathbb{Z}_4)$. Hence result follows. $\square$

Theorem 4.2. *Let*

$$a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, d = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}.$$

Then a presentation of $GL(2, \mathbb{Z}_{2^n})$, for $n > 2$ is

$$\begin{aligned} \langle a, b, c, d \mid a^2, b^2, c^4, d^{2^{n-1}}, (ab)^3, (bc)^2, (bd)^{2^{n-2}}, (bca)^{2^n}, \\ \{d^2, c^2\} \in \text{center}, d(bca)d^{-1} = (cab)^3, cbd = dbc, \\ dc^{-1}d^{-1} = (cab)^{-3}(bca)^{3^{-1}}(cab)^{-3} \rangle. \end{aligned}$$

Proof. Let G be a group generated by a, b, c, d and having presentation,

$$\begin{aligned} \langle a, b, c, d \mid a^2, b^2, c^4, d^{2^{n-1}}, (ab)^3, (bc)^2, (bd)^{2^{n-2}}, (bca)^{2^n}, \\ \{d^2, c^2\} \in \text{center}, d(bca)d^{-1} = (cab)^3, cbd = dbc, \\ dc^{-1}d^{-1} = (cab)^{-3}(bca)^{3^{-1}}(cab)^{-3} \rangle. \end{aligned}$$

First we shall show that group G is finite. Consider a subgroup H of G having a presentation is given by

$$H = \langle s, t \mid s^{2^n}, t^4, (st)^3 = t^2, (s^3ts^{3^{-1}}t)^3 = 1, (s^6ts^{3^{-1}}t)^2 = t^2 \rangle,$$

where $s = bca$ and $t = c^{-1}$. By using above relators, we can see that H is normal subgroup of G and H is isomorphic to $SL(2, \mathbb{Z}_{2^n})$ (see Theorem 2.1 and Theorem 2.4). So H is finite. Further we have,

$$G/H = \langle b, d \mid b^2, d^{2^{n-2}}, bd = db \rangle.$$

Therefore G is finite. Now Suppose $x = bc$, $y = bd$. Then by using above

relators, the order of x is 2 and the order of y is 2^{n-2} . Consider

$$K = \langle x, y \mid x^2, y^{2^{n-2}}, xy = yx \rangle.$$

Then K is a subgroup of G . Thus the canonical form of $K = \langle x^i y^j \mid i = 0, 1, 0 \leq j \leq 2^{n-3} \rangle$ and the order of $K = 2^{n-1}$. We can directly see that an arbitrary element of K has the form $\begin{pmatrix} 1 & 0 \\ 0 & \pm 3^j \end{pmatrix}$. Since $3^j \not\equiv \pm 1 \pmod{2^n}$, so $H \cap K = \{I_2\}$. Thus $G = HK$, Now by Theorem 2.3, $o(GL(2, \mathbb{Z}_{2^n})) = 2^{n-1} o(SL(2, \mathbb{Z}_{2^n})) = o(G)$, we get $G = GL(2, \mathbb{Z}_{2^n})$. Hence result follows. $\square$

Corollary 4.1. $GL(2, \mathbb{Z}_{2^n}) \cong SL(2, \mathbb{Z}_{2^n}) \rtimes \mathcal{U}(\mathbb{Z}_{2^n})$.

Theorem 4.3. Let p be an odd prime and

$$a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, d = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}.$$

Then a presentation of $GL(2, \mathbb{Z}_{p^n})$ is

$$\begin{aligned} \langle a, b, c, d \mid a^2, b^2, c^4, d^{2\phi(p^n)}, (ab)^3, (bc)^2, (bd)^{\phi(p^n)}, (bca)^{p^n}, cbd = dbc, \\ \{d^2, c^2\} \in \text{center}, (bc)(bd)^{\frac{\phi(p^n)}{2}}, d(bca)d^{-1} = (cab)^\alpha, \\ dc^{-1}d^{-1} = (cab)^{-\alpha}(bca)^{\alpha-1}(cab)^{-\alpha} \rangle. \end{aligned}$$

Proof. Suppose G is a group generated by a, b, c, d with a presentation,

$$\begin{aligned} \langle a, b, c, d \mid a^2, b^2, c^4, d^{2\phi(p^n)}, (ab)^3, (bc)^2, (bd)^{\phi(p^n)}, (bca)^{p^n}, cbd = dbc, \\ \{d^2, c^2\} \in \text{center}, (bc)(bd)^{\frac{\phi(p^n)}{2}}, d(bca)d^{-1} = (cab)^\alpha, \\ dc^{-1}d^{-1} = (cab)^{-\alpha}(bca)^{\alpha-1}(cab)^{-\alpha} \rangle. \end{aligned}$$

First we shall show that group G is finite. Consider a subgroup H of G having a presentation is given by

$$H = \langle s, t \mid s^{p^n}, t^4, (st)^3 = t^2, (s^\alpha t s^{\alpha-1} t)^3 = 1, (s^{2\alpha} t s^{\alpha-1} t)^2 = t^2 \rangle,$$

where $s = bca$ and $t = c^{-1}$. By using above relators, we can see that H is a normal subgroup of G and it is isomorphic to $SL(2, \mathbb{Z}_{p^n})$ (see Theorem 2.1 and Theorem 2.4). So H is finite. If $p \equiv 1 \pmod{4}$, then

$$G/H = \langle d \mid d^{\phi(p^n)} \rangle.$$

If $p \equiv -1 \pmod{4}$, then

$$G/H = \langle b, d \mid b^2, d^{\frac{\phi(p^n)}{2}}, bd = db \rangle.$$

Therefore G is finite.

Now Suppose $x = bd$. Then by using above relators, the order of x is $\phi(p^n)$. Consider

$$K = \langle x \mid x^{\phi(p^n)} \rangle.$$

Then K is a subgroup of G and the order of $K = \phi(p^n)$. We can directly see that an arbitrary element of K has the form $\begin{pmatrix} 1 & 0 \\ 0 & \pm \alpha^j \end{pmatrix}$, hence $H \cap K = \{I_2\}$. Thus $G = HK$. Now by Theorem 2.3, $o(GL(2, \mathbb{Z}_{p^n})) = \phi(p^n) o(SL(2, \mathbb{Z}_{p^n})) = o(G)$, we get $G = GL(2, \mathbb{Z}_{p^n})$. Hence result follows. $\square$

Corollary 4.2. *We have another presentation for $GL(2, \mathbb{Z}_{p^n})$ including three Lie regular generators which is given by*

$$\begin{aligned} \langle a, b, d \mid a^2, b^2, d^{2\phi(p^n)}, c = b(bd)^{\frac{\phi(p^n)}{2}}, (ab)^3, (bc)^2, (bd)^{\phi(p^n)}, (bca)^{p^n}, \\ cbd = dbc, \{d^2, c^2\} \in \text{center}, (bc)(bd)^{\frac{\phi(p^n)}{2}}, d(bca)d^{-1} = (cab)^\alpha, \\ dc^{-1}d^{-1} = (cab)^{-\alpha}(bca)^{\alpha-1}(cab)^{-\alpha} \rangle. \end{aligned}$$

Proof. Replace c by $b(bd)^{\frac{\phi(p^n)}{2}}$ in the above theorem. $\square$

Corollary 4.3. $GL(2, \mathbb{Z}_{p^n}) \cong SL(2, \mathbb{Z}_{p^n}) \rtimes \mathcal{U}(\mathbb{Z}_{p^n})$.

Corollary 4.4. *The quotient group $GL(2, \mathbb{Z}_{p^n})/SL(2, \mathbb{Z}_{p^n})$ is cyclic for $p \neq 2$.*

Conclusion In 1966, P. M. Cohn has obtained a presentation for $GL(2, \mathbf{R})$, where $\mathbf{R}$ is a local ring. Observe that $\mathbb{Z}_{p^n}$ is a local ring, so we have a presentation for $GL(2, \mathbb{Z}_{p^n})$ and the number of generators depends on the size of $\mathbb{Z}_{p^n}$ and $\mathcal{U}(\mathbb{Z}_{p^n})$ (see [1]). In this article, we obtain a presentation for $GL(2, \mathbb{Z}_{p^n})$ using at most four generators.

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**HERMITIAN TRIPLE SYSTEMS ASSOCIATED WITH
BILINEAR FORMS AND CERTAIN APPLICATIONS
TO FIELD THEORY**

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Abstract

We define hermitian triple systems and give several examples induced from certain bilinear forms. We apply such a system to a field theory and obtain a Chern-Simons gauge theory.

Keywords: generalized Jordan triple systems; Lie algebras; hermitian operators; 3-algebra; Chern-Simons gauge theories; M-theory.

1 Introduction

The field of nonassociative algebras is rich in algebraic structure. It provides an important common ground for various branches of mathematics and physics, not only for Jordan algebras, Lie algebras and triple systems, but also for applications to M-theory and field theory in physics. Thus we will discuss the concept of triple systems with a ternary product $\langle xyz \rangle$ for a study of nonassociative algebras, in particular.

The structure theory of the generalized Jordan triple systems of second order of finite dimensional or called Kantor triple systems, which contains the concept of Jordan algebras and Jordan triple systems have been studied by author or other mathematicians ([1-13]).

On the other hand, it is well known ([14-24]) that the symmetric bounded domains are a one to one, correspondence to hermitian Jordan triple systems, for which a certain trace form is positive definite hermitian and that the concept of triple systems is useful to construct of Lie algebras or superalgebras.

Hence as a generalization of these concept, it seems that we are interesting to investigate for structure theory of hermitian generalized Jordan triple systems, in particular, the case of hermitian generalized Jordan triple systems of second order.

We shall concerned with algebras and triple systems which are infinite over a complex number field, unless otherwise specified.

In the point of view of string theory, the hermitian generalized Jordan triple systems are generalizations of the hermitian 3-algebras ([25-67]), which have been played crucial roles in M-theory. The field theories applied with the hermitian 3-algebras are the Chern-Simons gauge theories that describe effective actions of membranes in M-theory. Moreover, 3-algebra models of M-theory itself were proposed and have been studied in ([68-71]). Therefore, it is interesting to apply the hermitian generalized Jordan triple systems to the field theories.

We defined hermitian generalized Jordan triple systems and hermitian (ϵ, δ) -Freudenthal-Kantor triple system and proved structure theorems in ([72,73]). In this paper, we give several examples of the triple systems induced from certain bilinear forms. We apply such a system to a field

theory and obtain a Chern-Simons gauge theory. We also give some other examples without generalized Jordan triple systems or (ϵ, δ) -Freudenthal-Kantor triple systems and examples of nonassociative algebras given by our triple systems.

2 Definition and preamble

In this section, we give several definitions and the preamble in order to make present paper as self-contained as possible.

Definition. A triple systems U is said to be a $*$ -generalized Jordan triple system of second order if relations (0)–(iv) satisfy;

- 0) U is a Banach space,
 - i) $[L(a, b), L(c, d)] = L(\langle abc \rangle, d) - L(c, \langle bad \rangle)$,
 - ii) $K(\langle abc \rangle, d) + K(c, \langle abd \rangle) + K(a, K(c, d)b) = 0$,
- where $L(a, b)c = \langle abc \rangle$ and $K(a, b)c = \langle acb \rangle - \langle bca \rangle$,
- iii) $\langle xyz \rangle$ is $\mathbf{C}$ -linear operator on x, z and $\mathbf{C}$ -anti-linear operator on y ,
 - iv) $\langle abc \rangle$ continuous with respect to a norm $\| \cdot \|$ that is, there exists $K > 0$ such that

$$\| \langle xxx \rangle \| \leq K \|x\|^3 \text{ for all } x \in U.$$

Furthermore, a $*$ -generalized Jordan triple system of second order is said to be hermitian if it satisfies the following condition,

- v) all operator $L(x, y)$ is a positive hermitian operator with a hermitian metric

$$(x, y) = \text{trace } L(x, y),$$

that is, $(L(x, y)z, w) = (z, L^*(x, y)w)$, and $(x, y) = \overline{(y, x)}$.

In particular, if a triple system U satisfies the conditions (0),(i),(iii),(iv) and (v), then it is said to be a hermitian generalized Jordan triple system.

Furthermore, as a generalization, it is said to be a (ϵ, δ) -Freudenthal-Kantor triple system if the following relations satisfy;

- i)' $[L(a, b), L(c, d)] = L(\langle abc \rangle, d) + \epsilon L(c, \langle bad \rangle)$,
 - ii)' $K(\langle abc \rangle, d) + K(c, \langle abd \rangle) + \delta K(a, K(c, d)b) = 0$,
- where $K(a, b)c = \langle acb \rangle - \delta \langle bca \rangle$, $\epsilon = \pm 1, \delta = \pm 1$.

For (ε, δ) Freudenthal-Kantor triple systems, its (v) is generalized as follows,

$$(v)' \quad (L(x, y)z, w) = -\varepsilon(z, L^*(x, y)w), \text{ and } (x, y) = -\varepsilon(\overline{y}, \overline{x}).$$

Also, in particular, it is said to be a (ε, δ) Jordan triple system if $K(x, y) = 0$ for all $x, y \in U$.

We note that the concept of generalized Jordan triple system of second order coincides with that of $(-1, 1)$ -Freudenthal-Kantor triple system.

Example 1.1 Let U be a J^* -triple (for the definition, to see [15]). Then, U is a hermitian generalized Jordan triple system of second order, because $K(a, b)c = \langle abc \rangle - \langle cba \rangle$ and $K(a, b)c \equiv 0$ (identically zero).

Example 1.2. Let $T_{n,n}^*$ be the space of diagonal matrix of $n \times n$ with element of the complex number. Then $T_{n,n}^*$ is a $*$ -generalized Jordan triple system of second order, with respect to the product and the norm $\| \cdot \|$

$$\langle xyz \rangle = x\bar{y}^T z + z\bar{y}^T x - \bar{y}x^T z$$

for all $x, y, z \in T_{n,n}^*$

$$\|x\| = \max(|\lambda_i|, |\mu_i|)$$

where $x = \sum(\lambda_i e_i + \mu_i \sqrt{-1} e_i)$, and e_i are matrix unit element of $T_{n,n}^*$ and x^T is the transpose of x .

Remark We note that if $(U, \langle xyz \rangle)$ is a (ε, δ) Freudenthal-Kantor triple system with an endomorphism P of U s.t. $P^2 = Id$, and $P \langle xyz \rangle = \langle PxPyPz \rangle$, then new triple product defined by $\{xyz\} = \langle xPyz \rangle$ is a (ε, δ) Freudenthal-Kantor triple system.

Also, we have an example of P as follows:

For any $x = (x_1, \dots, x_n) \in U$, we define

$$P(x) = \bar{x} := (\bar{x}_1, \dots, \bar{x}_n) \in U,$$

where $\dim_{\mathbb{C}} U = n$ and all $\bar{x}_i$ denotes the conjugation of the complex number $\mathbb{C}$, then we have

$$P \langle xyz \rangle = \langle PxPyPz \rangle \text{ and } P^2 = Id.$$

Let U be a $*$ -generalized Jordan triple system. Then we may define the notation of tripotent and bitripotent as follows.

Definition. It is said to be a tripotent of U if

$$\langle ccc \rangle = c, \quad c \in U.$$

Definition. It is said to be a bitripotent of U if a pair (c_1, c_2) of the tripotents satisfy the relations

$$\langle c_1 c_1 c_2 \rangle = \alpha c_2, \quad \langle c_2 c_2 c_1 \rangle = \alpha c_1,$$

$$\langle c_1 c_2 c_1 \rangle = \beta c_2, \quad \langle c_2 c_1 c_2 \rangle = \beta c_1,$$

$$\langle c_2 c_1 c_1 \rangle = \gamma c_2, \quad \langle c_1 c_2 c_2 \rangle = \gamma c_1,$$

and other products are zero, where

$$\alpha^2 + \beta^2 + \gamma^2 \neq 0, \quad \alpha, \beta, \gamma \in \mathbf{R}. (\text{real number})$$

3 Several examples

In this section, we give several examples induced from certain bilinear forms.

Example 2.1 Let U be a vector space equipped with a bilinear form $\langle x, y \rangle$ such that $\langle x, y \rangle = \overline{\langle y, x \rangle}$, $\langle x, \bar{y} \rangle = \langle y, \bar{x} \rangle$ and $\langle \alpha x, \beta y \rangle = \alpha \bar{\beta} \langle x, y \rangle$. Then

$$\langle xyz \rangle = \langle x, y \rangle z + \langle z, y \rangle x - \langle z, \bar{x} \rangle \bar{y}.$$

is a hermitian $(-1, 1)$ Jordan triple system.

Example 2.2 Let U be a vector space equipped with a bilinear form $\langle x, y \rangle$ such that $\langle x, y \rangle = -\overline{\langle y, x \rangle}$, $\langle x, \bar{y} \rangle = -\langle y, \bar{x} \rangle$ and $\langle \alpha x, \beta y \rangle = \alpha \bar{\beta} \langle x, y \rangle$. Then

$$\langle xyz \rangle = \langle x, y \rangle z - \langle z, y \rangle x - \langle z, \bar{x} \rangle \bar{y}.$$

is a hermitian $(1, -1)$ Jordan triple system.

Example 2.3 Let U be a vector space equipped with a bilinear form $\langle x, y \rangle$ such that $\langle x, y \rangle = \overline{\langle y, x \rangle}$, $\langle x, \bar{y} \rangle = \langle y, \bar{x} \rangle$ and $\langle \alpha x, \beta y \rangle = \alpha \bar{\beta} \langle x, y \rangle$. Then

$$\langle xyz \rangle = \langle x, y \rangle z - \langle z, y \rangle x.$$

is a hermitian $(-1, -1)$ Jordan triple system.

Example 2.4 Let U be a vector space equipped with a bilinear form $\langle x, y \rangle$ such that $\overline{\langle x, y \rangle} = \langle \tilde{x}, \tilde{y} \rangle = \langle y, x \rangle$, $\langle x, \tilde{y} \rangle = -\langle y, \tilde{x} \rangle$ and $\langle \alpha x, \beta y \rangle = \alpha \bar{\beta} \langle x, y \rangle$ with $\tilde{\tilde{x}} = -x$. Then

$$\langle xyz \rangle = -\langle x, y \rangle z + \langle z, y \rangle x - \langle z, \tilde{x} \rangle \tilde{y}.$$

is a hermitian $(-1, -1)$ Jordan triple system. If we define a bilinear form as $\langle x, y \rangle = -x_1 \cdot \bar{y}_1 - x_2 \cdot \bar{y}_2$ for $x = (x_1, x_2)$ and $\tilde{x} = (\bar{x}_2, -\bar{x}_1)$, where x_1 and x_2 are N -vectors, this system is called $sp(2N) \oplus u(1)$ hermitian 3-algebra in M -theory [39, 41, 42].

Example 2.5 Let U be a vector space equipped with a bilinear form $\langle x, y \rangle$ such that $\langle x, y \rangle = -\overline{\langle y, x \rangle}$, $\langle x, \bar{y} \rangle = -\langle y, \bar{x} \rangle$ and $\langle \alpha x, \beta y \rangle = \alpha \bar{\beta} \langle x, y \rangle$. Then

$$\langle xyz \rangle = \langle x, y \rangle z + \langle z, y \rangle x.$$

is a hermitian $(1, 1)$ Jordan triple system.

Next we consider the case of $K(x, y) \neq 0$.

Proposition 2.6 Let U be a vector space equipped with a bilinear form $\langle x, y \rangle$ such that $\langle x, y \rangle = -\varepsilon \overline{\langle y, x \rangle}$, $\langle x, \bar{y} \rangle = -\varepsilon \langle y, \bar{x} \rangle$ and $\langle \alpha x, \beta y \rangle = \alpha \bar{\beta} \langle x, y \rangle$. Then

$$\langle xyz \rangle = \langle z, y \rangle x.$$

is a hermitian (ε, δ) Freudenthal-Kantor triple system.

Proposition 2.7 Let U be a vector space equipped with a bilinear form $\langle x, y \rangle$ such that $\langle x, y \rangle = -\overline{\langle y, x \rangle}$, $\langle x, \bar{y} \rangle = -\langle y, \bar{x} \rangle$ and $\langle \alpha x, \beta y \rangle = \alpha \bar{\beta} \langle x, y \rangle$. Then

$$\langle xyz \rangle = 1/2(-\langle x, y \rangle z - \langle z, y \rangle x + \langle x, \bar{z} \rangle \bar{y}).$$

is a hermitian $(1, 1)$ Freudenthal-Kantor triple system with $K(x, z)y = \langle x, \bar{z} \rangle \bar{y}$.

Proposition 2.7 Let U be a vector space equipped with a bilinear form $\langle x, y \rangle$ such that $\langle x, y \rangle = \overline{\langle y, x \rangle}$, $\langle x, \bar{y} \rangle = \langle y, \bar{x} \rangle$ and $\langle \alpha x, \beta y \rangle = \alpha \bar{\beta} \langle x, y \rangle$. Then

$$\langle xyz \rangle = 1/2(\langle x, y \rangle z - \langle z, y \rangle x + \langle x, \bar{z} \rangle \bar{y}).$$

is a hermitian $(-1,-1)$ Freudenthal-Kantor triple system with $K(x,z)y = \langle x, \bar{z} \rangle y$.

Proposition 2.8 ([11]) *Let U be a $(-1,1)$ Freudenthal-Kantor triple system with a left unit unit element u (i.e, $L(u,u)=Id$). Then we have a decomposition,*

$$U = U_1 \oplus U_3$$

where $U_i = \{x \mid \langle xuu \rangle = ix\}$.

Furthermore, we have

$$U_1 = U_1^+ \oplus U_1^-, \quad U_3 = U_3^+ \oplus U_3^-,$$

where

$$Q(x) := \langle uxu \rangle = \pm x \text{ if } x \in U_1^\pm, Q(x) := \langle uxu \rangle = \pm 3x \text{ if } x \in U_3^\pm.$$

Conversely, if we set $x \cdot y = \langle xuy \rangle$, then the triple product $\langle xyz \rangle$ can be expressed in terms of $x \cdot y$;

$$\langle xyz \rangle = (Q^{-1}(y) \cdot x) \cdot z + x \cdot (Q^{-1}(y) \cdot z) - Q^{-1}(y) \cdot (x \cdot z),$$

Remark We note that for accurate example of above bilinear forms,

$$\langle x, y \rangle = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n \Rightarrow \overline{\langle x, y \rangle} = \langle y, x \rangle.$$

where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$,

or

$$\langle x, y \rangle = x_1 \bar{y}_2 - x_2 \bar{y}_1 + \cdots + x_{2n-1} \bar{y}_{2n} - x_{2n} \bar{y}_{2n-1} \Rightarrow \overline{\langle x, y \rangle} = - \langle y, x \rangle,$$

where $x = (x_1, \dots, x_{2n})$, $y = (y_1, \dots, y_{2n})$.

Let U be a set of $Mat(1, n; \mathbf{C})$ for their triple products induced in this section and $x = \sum_i (\lambda_i + \mu_i \sqrt{-1}) e_i \in U$, where λ_i, μ_i are real numbers, if we set

$$\|x\| = \max(|\lambda_i|, |\mu_i|),$$

then we have a hermitian property, because $\|\langle xxx \rangle\| \leq K\|x\|^3$ and for $\varepsilon = \pm 1$ it holds $L(x, y)^* = -\varepsilon L(y, x)$ and $(L(x, y)z, w) = (z, L(x, y)^*w)$.

For a special example of Proposition 2.6 ($\varepsilon = -1, \delta = 1$ and $\dim_{\mathbf{C}} U = n$), we have a tripotent element as follows:

If e_i and e_j are basis of U satisfying $\langle e_i, e_j \rangle = \delta_{ij}$, then e_i and e_j are bitripotents of U , because $\langle e_j e_i e_i \rangle = e_j, \langle e_i e_j e_i \rangle = \langle e_i e_i e_j \rangle = \delta_{ij} e_i$.

4 Other triple systems

In this section, we give another examples without generalized Jordan triple systems and examples of nonassociative algebras given by our triple systems, furthermore exhibit an image of geometrical view point.

Example 3.1

Let U be a vector space equipped with a bilinear form $\langle x, y \rangle$ such that $\langle x, y \rangle = \overline{\langle y, x \rangle}$, $\langle x, \bar{y} \rangle = \langle y, \bar{x} \rangle$ and $\langle \alpha x, \beta y \rangle = \alpha \bar{\beta} \langle x, y \rangle$. Then

$$\langle xyz \rangle = \langle z, y \rangle x + \langle z, \bar{x} \rangle \bar{y}.$$

is a hermitian triple system satisfying the relation

$$\langle ab \langle xyz \rangle \rangle = \langle \langle abx \rangle yz \rangle + \langle x \langle bay \rangle z \rangle - \langle xy \langle abz \rangle \rangle.$$

Example 3.2

Let U be a vector space equipped with a bilinear form $\langle x, y \rangle$ such that $\langle x, y \rangle = -\overline{\langle y, x \rangle}$, $\langle x, \bar{y} \rangle = -\langle y, \bar{x} \rangle$ and $\langle \alpha x, \beta y \rangle = \alpha \bar{\beta} \langle x, y \rangle$. Then

$$\langle xyz \rangle = \langle z, y \rangle x - \langle z, \bar{x} \rangle \bar{y}$$

is a hermitian triple system satisfying the relation

$$\langle ab \langle xyz \rangle \rangle = \langle \langle abx \rangle yz \rangle - \langle x \langle bay \rangle z \rangle - \langle xy \langle abz \rangle \rangle.$$

Remark 3.3 Let U be a vector space over $\mathbf{C}$ with a hermitian bilinear form $\langle x, y \rangle = \overline{\langle y, x \rangle}$, $\langle \alpha x, \beta y \rangle = \alpha \bar{\beta} \langle x, y \rangle$ and an element e such that $\langle e, e \rangle = 1$ and $\bar{e} = e \in U$

If we define the binary product by

$$x \circ y = \langle y, e \rangle x$$

then binary algebra $(U, x \circ y)$ is a noncommutative Jordan algebra satisfying the flexible law, i.e.,

$$\begin{aligned} (x \circ y) \circ x &= x \circ (y \circ x) \\ x \circ (y \circ x^2) &= (x \circ y) \circ x^2 \end{aligned}$$

and have a right unit element , i.e.,

$$e \circ e = e, \quad x \circ e = x \quad \text{but} \quad e \circ x \neq x.$$

Indeed, we calculate

$$(x \circ y) \circ x = (< y, e > x) \circ x = < y, e > < x, e > x.$$

On the other hand, we have

$$x \circ (y \circ x) = x \circ (< x, e > y) = < x, e > < y, e > x.$$

Hence we obtain

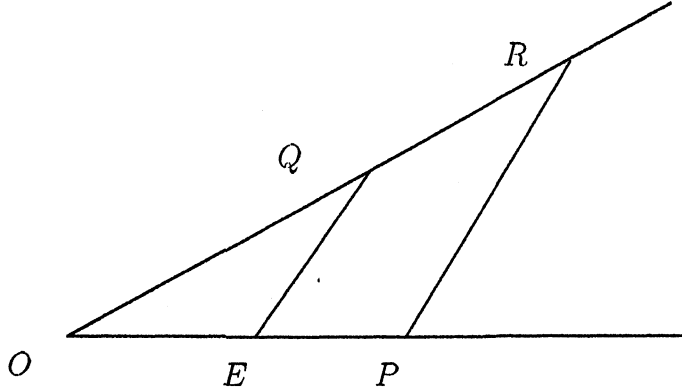
$$x \circ (y \circ x) = (x \circ y) \circ x.$$

Furthermore, by straight forward calculations, we have

$$(x \circ x) \circ y = x \circ (x \circ y), \quad (y \circ x) \circ x = y \circ (x \circ x), \quad (\text{alternative law})$$

$$(x, y, x^2) = 0 \quad (\text{Jordan property})$$

In geometrical view point roughly, we may make an image as follows:



$$\frac{|\overrightarrow{OR}|}{|\overrightarrow{OQ}|} = \frac{|\overrightarrow{OP}|}{|\overrightarrow{OE}|} \implies \overrightarrow{OR} = < y, e > x,$$

where $\overrightarrow{OQ} = x$, $\overrightarrow{OE} = e$, $|\overrightarrow{OE}| = 1$, $\overrightarrow{OP} = y$.

Indeed, from the fact that $|\overrightarrow{OP}| |\overrightarrow{OE}| = < y, e >$, $< e, e > = 1$, it follows the result.

5 Application to field theory

In this section, we apply a hermitian triple system to a field theory.

We start with

$$S = \int d^3x \text{tr}(-L(\mathbf{D}_\mu Z, \mathbf{D}^\mu Z) + V(Z) + \epsilon^{\mu\nu\lambda}(-\frac{1}{2}A_{\mu bc}\partial_\nu A_{\lambda da}L(<T^c T^b T^a>, T^d) + \frac{1}{3}A_{\mu da}A_{\nu bc}A_{\lambda fe}L(<T^c T^b T^a>, <T^f T^e T^d>))), (1)$$

where

$$\mathbf{D}_\mu Z = \partial_\mu Z - A_{\mu ba} <T^a T^b Z>. \quad (2)$$

Z and A_μ are matter and gauge fields, respectively. $V(Z)$ is a potential term. This action is invariant under the transformations generated by the operator $L(x, y) - L(y, x)$. The action (1) equipped with a Lorentzian Lie 3-algebra and a hermitian 3-algebra describes the bosonic parts of the effective actions of supermembranes in M-theory. We apply a hermitian triple system, $<xyz> = (x \cdot \bar{y})z + (z \cdot \bar{y})x - (z \cdot x)\bar{y}$ in Example 2.1 with a bilinear form $<x, y> = (x \cdot \bar{y})$ for vectors x and y , to this action and see what kind of field theory we obtain. The metric is given by $\text{tr}L(x, y) = (x \cdot \bar{y})$ after an appropriate renormalization.

The covariant derivative is explicitly written down as

$$\mathbf{D}_\mu Z = \partial_\mu Z - iA_\mu^L Z - iZ A_\mu^R, \quad (3)$$

where $A_\mu^L := -iA_{\mu ba}(T^a \cdot \bar{T}^b)$ generate the $u(1)$ Lie algebra, whereas $A_\mu^R := -iA_{\mu ba}(\bar{T}^b T^a - T^a \bar{T}^b)$. In order to see how gauge transformations act on these fields, let us see how $L(x, y) - L(y, x)$ acts on the coupling between A_μ and Z :

$$\begin{aligned} & \delta(A_{\mu ba} <T^a T^b Z>) \\ &= \Lambda_{dc} A_{\mu ba} (<T^c T^d T^a> T^b Z - <T^a <T^d T^c T^b> Z > \\ & \quad + <T^a T^b <T^c T^d Z>>) \\ &= -A_\mu^L (\Lambda^L Z) - A_\mu^L (Z \Lambda^R) + Z [\Lambda^R, A_\mu^R] - (\Lambda^L Z + Z \Lambda^R) A_\mu^R, \end{aligned} \quad (4)$$

where gauge parameters Λ^R and Λ^L are defined in the same way as A_μ^R and A_μ^L , respectively. From this, we obtain gauge transformations,

$$\begin{aligned}\delta A_\mu^R &= \partial_\mu \Lambda^R + i[A_\mu^R, \Lambda^R] \\ \delta A_\mu^L &= \partial_\mu \Lambda^L\end{aligned}\tag{5}$$

$$\delta Z = i\Lambda^L Z + Z(i\Lambda^R).\tag{6}$$

Therefore, A_μ^R and A_μ^L transform as adjoint representations, whereas Z transforms as a bi-fundamental representation. As a result, the action can be rewritten in a covariant form and we obtain a Chern-Simons gauge theory,

$$\begin{aligned}S &= \int d^3x \text{tr}(-(\partial_\mu Z - iA_\mu^L Z - iZ A_\mu^R) \cdot \overline{(\partial_\mu Z - iA_\mu^L Z - iZ A_\mu^R)} \\ &\quad + \epsilon^{\mu\nu\lambda}(\frac{1}{4}(2A_\mu^L \partial_\nu A_\lambda^L + A_\mu^R \partial_\nu A_\lambda^R) - \frac{i}{6}A_\mu^R A_\nu^R A_\lambda^R)).\end{aligned}\tag{7}$$

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FREE MODULES OVER ASSOCIATIVE
 n -TUPLE RINGS

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Abstract

In this paper, we study a notion of free module over rings having n bilinear mappings $\boxed{s} : R \times R \longrightarrow R$ ($s = 1, \dots, n$), satisfying the condition $(a\boxed{r}b)\boxed{s}c = a\boxed{r}(b\boxed{s}c)$ for all $a, b, c \in R$ and for each $r, s = 1, \dots, n$, called associative n -tuple rings. The notion of ring used in this paper is similar to a notion originally presented by N. A. Koreshkov in [1]. We obtain similar results for the classical free module theory.

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1 Associative n -tuple rings

In this section, we define the notions of associative n -tuple rings, division rings and the notion of modules on such rings, according to a similar concept presented in [1] for the class of associative n -tuple algebras. These modules are generalizations of the classical notions of free modules from the generalization of the notion of ring. Examples of such rings are given.

Definition 1.1. Let R be an abelian additive group and

$$\boxed{s} : R \times R \longrightarrow R \quad (s = 1, \dots, n)$$

n bilinear mappings. We say that R is an associative n -tuple ring if

$$(a\boxed{r}b)\boxed{s}c = a\boxed{r}(b\boxed{s}c)$$

for all $a, b, c \in R$ and for each $r, s = 1, \dots, n$. We call the elements of R scalars and we denote by $\mathcal{O}(R) = \{\boxed{1}, \dots, \boxed{n}\}$ the set of n bilinear mappings on R . An additive subgroup S of R is said to be a subring if $S\boxed{s}S \subseteq S$ for each $s = 1, \dots, n$. An additive subgroup I of R is said to be a left ideal (resp., right ideal) if $R\boxed{s}I \subseteq I$ (resp., $I\boxed{s}R \subseteq I$) for each $s = 1, \dots, n$. An additive subgroup I of R is said to be a two-sided ideal of R if I is a left ideal that is also a right ideal.

Let R be an associative n -tuple ring. We say that R is commutative if the n bilinear mappings $\boxed{1}, \dots, \boxed{n}$ are commutative, that is, satisfy $a\boxed{s}b = b\boxed{s}a$ for all a, b in R and for each $s = 1, \dots, n$. A nonzero element $1_s \in R$ ($1 \leq s \leq n$) is called a s -identity element if $1_s\boxed{s}a = a\boxed{s}1_s = a$ for all $a \in R$. We say that R is unitary if R has a s -identity element for each $s = 1, \dots, n$.

Let R be a unitary associative n -tuple ring. An element $a \in R$ is called s -left-invertible (resp. s -right-invertible) if there exists an element $b_s \in R$ (resp. there exists an element $c_s \in R$) such that $b_s\boxed{s}a = 1_s$ (resp. $a\boxed{s}c_s = 1_s$). In this case, the element b_s (resp. c_s) is said to be the s -left inverse element (resp. s -right inverse element) of a . An element $a \in R$ is called s -invertible if it is s -left-invertible and s -right-invertible. Certainly, if an element $a \in R$ is both a s -left and s -right invertible, then the s -left and

s-right inverse elements are equal. In this case, the element *s*-left inverse element or *s*-right inverse element of *a* is called a *s*-inverse element. We say that *R* is a division ring if *R* is nonzero and every nonzero element is *s*-invertible for each $s = 1, \dots, n$.

Let $M_m(R)$ be the matrix ring over an associative ring *R* (the set of $m \times m$ matrices with entries from *R*) and $c_1, c_2, \dots, c_n$ a finite family of matrices in $M_m(R)$. Let $\boxed{s} : M_m(R) \times M_m(R) \longrightarrow M_m(R)$ ($s = 1, \dots, n$) be *n* bilinear mappings defined by $a\boxed{s}b = ac_sb$ ($s = 1, \dots, n$). Then $M_m(R)$ is an associative *n*-tuple ring.

Let $H = Ri + Rj + Rk$ with the usual multiplications defined by $i^2 = -1, j^2 = -1$, and $ij = -ji = k$. Then *H* is a 4-dimensional division algebra over *R*. If $\alpha = a + bi + cj + dk$ where $a, b, c, d \in R$, then the element $\bar{\alpha} = a - bi - cj - dk$ satisfies $\alpha\bar{\alpha} = \bar{\alpha}\alpha = a^2 + b^2 + c^2 + d^2$. Moreover, if $\alpha \neq 0$, then its usual inverse is $\alpha^{-1} = (a^2 + b^2 + c^2 + d^2)^{-1}\bar{\alpha}$. Let $\gamma_1, \gamma_2, \dots, \gamma_n$ be distinct nonzero elements of *H*. Define news *n* bilinear mappings in *H* by the rules $\alpha\boxed{s}\beta = \alpha\gamma_s\beta$ ($s = 1, \dots, n$). Then, *H* is a noncommutative division ring, where the nonzero elements $\gamma_s^{-1} (= 1/\gamma_s)$ are the *s*-identity elements of *H* and $\gamma_s^{-1}\alpha^{-1}\gamma_s^{-1}$ is the *s*-inverse element of α for each $s = 1, \dots, n$.

Definition 1.2. Let *R* be an associative *n*-tuple ring. An additive group *M* is said to be a left module over *R* if there are *n* bilinear mappings

$$*_s : R \times M \longrightarrow M \quad (s = 1, \dots, n),$$

called actions of *R* on *M*, where $\boxed{r}$ and $*_r$ are related in such a way that

$$(a\boxed{r}b) *_s m = a *_r (b *_s m)$$

for all $a, b \in R, m \in M$ and for each $r, s = 1, \dots, n$. We denote by $\mathcal{O}(M) = \{*_1, \dots, *_n\}$ the set of *n* actions of *R* on *M*. The left *R*-module *M* is said to be unitary if *R* is a unitary *n*-tuple ring satisfying the condition $1_s *_s m = m$ for all element $m \in M$ and for each $s = 1, \dots, n$.

A subgroup *L* of a left *R*-module *M* is said to be a submodule if the *n* actions of *R* on *M* satisfy the condition $R *_s L \subseteq L$ for each $s = 1, \dots, n$.

Every R -module M contains at least two submodules, (0) and M , which are called trivial. A submodule is called proper submodule if it is not trivial. A nonzero R -module M is called simple if it has no proper submodule.

It is evident that every associative n -tuple ring R can be considered as a left R -module over itself, taking the left action of $a \in R$ on $b \in R$ given by $r *_s m = r \boxed{s} m$.

A linear mapping φ of an left R -module M into an left R -module M' is said to be a R -homomorphism if

$$\varphi(a *_s m) = a *_s' \varphi(m)$$

where the bilinear mappings $*$ and $*_s'$ correspond to the bilinear mapping $\boxed{s}$, for all $a \in R$, $m \in M$ and for each $s = 1, \dots, n$.

We say that φ is a R -monomorphism (resp., R -epimorphism) if the mapping φ is injective (resp., surjective). We say that φ is a R -isomorphism if the mapping φ is a R -monomorphism and an R -epimorphism. We say that M is R -isomorphic to M' if there is a R -isomorphism $\varphi : M \rightarrow M'$. In this case, we denote this relation by $M \simeq M'$.

The quotient module of a left R -module M by a submodule N is the quotient group $M/N = \{m + N, m \in M\}$, where the action of R on M/N is defined by the rule

$$a *_s (m + N) = (a *_s m) + N,$$

for all $a \in R$, $m \in M$ and for each $s = 1, \dots, n$. An analog of the first homomorphism theorem for modules is obvious. Namely, let R be a n -tuple ring and M and M' left R -modules. If $\varphi : M \rightarrow M'$ is a R -epimorphism, then M' is isomorphic to the quotient module $M/\text{Ker}(\varphi)$, where $\text{Ker}(\varphi) = \{m \in M \mid \varphi(m) = 0\}$ is a submodule of M .

Definition 1.3. Let X be a set and I an arbitrary set of indices. We define the set $X^I = \{x \mid x : I \rightarrow X \text{ is a mapping}\}$. We call the elements of X^I families of elements of X and we denote any family x by $(x_i)_{i \in I}$, where $x(i) = x_i$, for all $i \in I$.

If R is a n -tuple ring, M a left R -module and I an arbitrary set of indices, then the set M^I is a left R -module with addition and the n actions

of R on M^I defined by

$$(m_i)_{i \in I} + (m'_i)_{i \in I} = (m_i + m'_i)_{i \in I} \text{ and } a *_s (m_i)_{i \in I} = (a *_s m_i)_{i \in I},$$

respectively, for all $(m_i)_{i \in I}, (m'_i)_{i \in I} \in M^{(I)}$, $a \in R$ and for each $s = 1, \dots, n$. A family $(m_i)_{i \in I}$ of elements of M is said to be almost-null if the set $\{i \in I : m_i \neq 0\}$ is finite. The set of the almost-null families of elements of M is a submodule of M^I and is denoted by $M^{(I)}$. Since any associative n -tuple ring R is a left R -module over itself, then for any set of indices I the set R^I is a left R -module and $R^{(I)}$ is a submodule of R^I , where we observe that the n actions of R on R^I , in this case, are given by $a *_s (\lambda_i)_{i \in I} = (a \square_s \lambda_i)_{i \in I}$, for all $a \in R$, $(\lambda_i)_{i \in I} \in R^I$ and for each $s = 1, \dots, n$.

Let R be a n -tuple ring and M a left R -module. If $(M_i)_{i \in I}$ is a family of submodules of M , then we define the left R -module

$$\sum_{i \in I} M_i = \left\{ \sum_{i \in I} m_i \mid (m_i)_{i \in I} \in M^{(I)} \text{ with } m_i \in M_i, \text{ for each } i \in I \right\},$$

called sum of the family $(M_i)_{i \in I}$. We say that this sum is a direct sum if every element of $\sum_{i \in I} M_i$ can be written in a unique way. In this case, we write the sum of the family $(M_i)_{i \in I}$ as $\oplus_{i \in I} M_i$.

For more details on the concepts developed by N. A. Koreshkov, see [1].

Throughout this paper all the associative n -tuple rings and left R -modules are considered unitary and we refer to M simply as a R -module.

2 Free modules on associative n -tuple rings

In this section, we define the notions of linear combinations, free modules over a set, free modules over associative n -tuple rings, linear independence and basis. We study some of their properties, relations and consequences. We obtain similar results with respect to the classical free module theory. The results obtained in this section are inspired by the ideas presented in [2].

Definition 2.1. Let R be a n -tuple ring, M a R -module and $(m_i)_{i \in I}$ a family of the elements of M . For any almost-null family $(\lambda_i)_{i \in I}$ of elements of R and any family $(*)_{s_i}^{i \in I}$ of elements of $\mathcal{O}(M)$ the element of M defined by

$$\sum_{i \in I} \lambda_i *_{s_i} m_i$$

is called a linear combination of the family $(m_i)_{i \in I}$.

Proposition 2.2. Let R be a n -tuple ring, M a R -module and $(m_i)_{i \in I}$ a family of the elements of M . Then the set of all linear combinations of the family $(m_i)_{i \in I}$ is the smallest submodule of M that contains the family $(m_i)_{i \in I}$. It is called the submodule generated by family $(m_i)_{i \in I}$ and denoted by $\langle (m_i)_{i \in I} \rangle$.

Proof. Since $(0)_{i \in I}$ is a almost-null family of elements of R and $0 = \sum_{i \in I} 0 *_{s_i} m_i$, for any family $(*)_{s_i}^{i \in I}$ of elements of $\mathcal{O}(M)$, then 0 is a linear combination of the family $(m_i)_{i \in I}$. Let a be a scalar, $\boxed{s} \in \mathcal{O}(R)$, $(\lambda_i)_{i \in I}$, $(\mu_i)_{i \in I}$ two almost-null families of elements of R and $(*)_{r_i}^{i \in I}$, $(*)_{s_i}^{i \in I}$ two families of elements of $\mathcal{O}(M)$. Then

$$\begin{aligned} & a \boxed{s} \left(\sum_{i \in I} \lambda_i *_{r_i} m_i \right) + \sum_{i \in I} \mu_i *_{s_i} m_i \\ &= a \boxed{s} \left(\sum_{i \in I} \lambda_i *_{r_i} (1_{s_i} *_{s_i} m_i) \right) + \sum_{i \in I} \mu_i *_{s_i} m_i \\ &= a \boxed{s} \left(\sum_{i \in I} (\lambda_i \boxed{r_i} 1_{s_i}) *_{s_i} m_i \right) + \sum_{i \in I} \mu_i *_{s_i} m_i \\ &= \sum_{i \in I} \left(a \boxed{s} (\lambda_i \boxed{r_i} 1_{s_i}) *_{s_i} m_i + \mu_i *_{s_i} m_i \right) \\ &= \sum_{i \in I} \left(a \boxed{s} (\lambda_i \boxed{r_i} 1_{s_i}) + \mu_i \right) *_{s_i} m_i. \end{aligned}$$

Since $(a \boxed{s} (\lambda_i \boxed{r_i} 1_{s_i}) + \mu_i)_{i \in I}$ is an almost-null family of elements of R , then we can conclude that the set of all linear combinations of the family

$(m_i)_{i \in I}$ is a submodule of M . It is clear that this submodule is the smallest submodule of M that contains the family $(m_i)_{i \in I}$. $\square$

Definition 2.3. Let R be a n -tuple ring and M a R -module. We say that a family $(m_i)_{i \in I}$ of the elements of M generates M if $M = \langle (m_i)_{i \in I} \rangle$. The R -module M is called finitely generated if there is a family $(m_i)_{i \in I}$ of the elements of M that generates M , where I is a finite set.

Definition 2.4. Let R be a n -tuple ring, I a non-empty set and M a R -module. We say that M is free on I if there is a mapping $f : I \longrightarrow M$ such that, for every R -module N and every mapping $g : I \longrightarrow N$ there is a unique R -homomorphism $h : M \longrightarrow N$ such that the diagram

$$\begin{array}{ccc} I & \xrightarrow{g} & N \\ f \downarrow & \nearrow h & \\ M & & \end{array}$$

is commutative. We denote such R -module by (M, f) .

In the next two results, the proofs are similar to those found in [2], so we omit them here.

Theorem 2.5. Let R be a n -tuple ring, I a non-empty set and M a R -module. If (M, f) is free on I then f is injective and the family $(f(i))_{i \in I}$ generates M .

Theorem 2.6. [Uniqueness] Let R be a n -tuple ring, I a non-empty set and M, M' R -modules such that (M, f) is free on I . Then (M', f') is also a free R -module on I if and only if there is a unique R -isomorphism $\varphi : M \longrightarrow M'$ such that $\varphi \circ f = f'$.

Theorem 2.7. [Existence] Let R be a n -tuple ring and I a non-empty set. There exists a free R -module on I .

Proof. Let's fix a family $(\overline{s_i})_{i \in I}$ of elements of $\mathcal{O}(R)$ and take its corresponding family $(*)_{s_i}^*$ of elements of $\mathcal{O}(M)$. Let's define the family $f : I \longrightarrow R^{(I)}$ by assigning to every $i \in I$ the mapping $f(i) : I \longrightarrow R$ given by

$$[f(i)](j) = \begin{cases} 1_{s_i} & j = i \\ 0 & j \neq i \end{cases}.$$

Then, $(R^{(I)}, f)$ is a free R -module on I . Indeed, suppose that N is a R -module and that $g : I \longrightarrow N$ is a mapping. Define a mapping $h : R^{(I)} \longrightarrow N$ by

$$h(\lambda) = \sum_{i \in I} \lambda_i *_{s_i} g(i)$$

for all $\lambda \in R^{(I)}$. It is clear that h is a R -homomorphism. Moreover, since $(h \circ f)(i) = h(f(i)) = \sum_{j \in I} [f(i)](j) *_{s_j} g(j) = g(i)$, for all $i \in I$, then $h \circ f = g$. To establish the uniqueness of the R -homomorphism h we observe that for any $\lambda \in R^{(I)}$, we have $\lambda_i = \lambda_i [s_i] 1_{s_i} = \sum_{j \in I} \lambda_j [s_j] [f(j)](i) = \left(\sum_{j \in I} \lambda_j *_{s_j} f(j) \right) (i)$, for all $i \in I$, and so $\lambda = \sum_{j \in I} \lambda_j *_{s_j} f(j)$. Thus, if $h' : R^{(I)} \longrightarrow M$ is also a R -homomorphism such that $h' \circ f = g$ then

$$\begin{aligned} h'(\lambda) &= h' \left(\sum_{j \in I} \lambda_j *_{s_j} f(j) \right) = \sum_{j \in I} \lambda_j *_{s_j} h'(f(j)) = \sum_{j \in I} \lambda_j *_{s_j} g(j) \\ &= \sum_{j \in I} \lambda_j *_{s_j} h(f(j)) = h \left(\sum_{j \in I} \lambda_j *_{s_j} f(j) \right) = h(\lambda) \end{aligned}$$

for all $\lambda \in R^{(I)}$. Therefore $h' = h$. □

Definition 2.8. Let R be a n -tuple ring and M a R -module. We say that M is free if there is a free R -module (F, f) on some non-empty I such that M is R -isomorphic to F .

Definition 2.9. Let R be a n -tuple ring, M a R -module and $(m_i)_{i \in I}$ a family of elements of M . We say that the family $(m_i)_{i \in I}$ is linearly independent or free if for every almost-null family $(\lambda_i)_{i \in I}$ of elements of R and for every family $(*)_{i \in I}$ of elements of $\mathcal{O}(M)$ the relation

$$\sum_{i \in I} \lambda_i *_{s_i} m_i = 0 \text{ implies } \lambda_i = 0, \text{ for all } i \in I.$$

A set that is not linearly independent is said to be linearly dependent. We say that the family $(m_i)_{i \in I}$ is a basis of M if it is linearly independent and generates M .

Theorem 2.10. *Let R be a n -tuple ring, M a R -module and $(m_i)_{i \in I}$ a family of elements of M . Then, the family $(m_i)_{i \in I}$ is a basis of M if and only if every element of M can be written as a linear combination univocally determined by each family $(*)_{r_i}^{i \in I}$ of elements of $\mathcal{O}(M)$. Moreover, if $(\lambda_i)_{i \in I}, (\mu_i)_{i \in I}$ are two almost-null families of elements of R and $(*)_{r_i}^{i \in I}, (*)_{s_i}^{i \in I}$ are two families of elements of $\mathcal{O}(M)$ such that $\sum_{i \in I} \lambda_i *_{r_i} m_i = \sum_{i \in I} \mu_i *_{s_i} m_i$, then $\lambda_i = \mu_i [s_i] 1_{r_i}$ for all $i \in I$.*

Proof. If $(m_i)_{i \in I}$ is a basis of M , then each element $m \in M$ can be expressed as a linear combination of elements of $(m_i)_{i \in I}$. Let $(*)_{s_i}^{i \in I}$ be a family of elements of $\mathcal{O}(M)$ and suppose that $m = \sum_{i \in I} \lambda_i *_{r_i} m_i = \sum_{i \in I} \mu_i *_{r_i} m_i$, where $(\lambda_i)_{i \in I}, (\mu_i)_{i \in I}$ are two almost-null families of elements of R . Then, we have $\sum_{i \in I} (\lambda_i - \mu_i) *_{r_i} m_i = 0$ which implies $\lambda_i - \mu_i = 0$, for all $i \in I$, since $(m_i)_{i \in I}$ is linearly independent and therefore such expressions are univocally determined for each family $(*)_{r_i}^{i \in I}$ of elements of $\mathcal{O}(M)$. Moreover, if $(\lambda_i)_{i \in I}, (\mu_i)_{i \in I}$ are two almost-null families of elements of R and $(*)_{r_i}^{i \in I}, (*)_{s_i}^{i \in I}$ are two families of elements of $\mathcal{O}(M)$ such that $\sum_{i \in I} \lambda_i *_{r_i} m_i = \sum_{i \in I} \mu_i *_{s_i} m_i$, then $\sum_{i \in I} \lambda_i *_{r_i} m_i = \sum_{i \in I} \mu_i *_{s_i} m_i = \sum_{i \in I} \mu_i *_{s_i} (1_{r_i} *_{r_i} m_i) = \sum_{i \in I} (\mu_i [s_i] 1_{r_i}) *_{r_i} m_i$ which results $\lambda_i = \mu_i [s_i] 1_{r_i}$ for all $i \in I$. Reciprocally, suppose that the condition holds. Then clearly the family $(m_i)_{i \in I}$ generates M and the element 0 can be expressed uniquely as the linear combination null of the family $(m_i)_{i \in I}$ by each family $(*)_{r_i}^{i \in I}$ of elements of $\mathcal{O}(M)$. This results that it is also linearly independent and so is a basis of M . $\square$

Theorem 2.11. *Let R be a n -tuple ring and $(R^{(I)}, f)$ the free R -module on a non-empty set I , according to Theorem 2.7. Then the family $(f(i))_{i \in I}$ is a basis of $R^{(I)}$.*

Proof. First, note that the family $(f(i))_{i \in I}$ generates M , by Theorem 2.5. Now, if $(\lambda_i)_{i \in I}, (\mu_i)_{i \in I}$ are two almost-null families of elements of R and $(*)_{s_i}^{i \in I}$ is a family of elements of $\mathcal{O}(R^{(I)})$ such that $\sum_{i \in I} \lambda_i *_{s_i} f(i) = \sum_{i \in I} \mu_i *_{s_i} f(i)$,

then

$$\begin{aligned}\lambda_j &= \sum_{i \in I} \lambda_i [s_i] [f(i)](j) = \left(\sum_{i \in I} \lambda_i *_{s_i} f(i) \right) (j) \\ &= \left(\sum_{i \in I} \mu_i *_{s_i} f(i) \right) (j) = \sum_{i \in I} \mu_i [s_i] [f(i)](j) = \mu_j\end{aligned}$$

for all $j \in I$. Thus, every element of $R^{(I)}$ can be written as a linear combination univocally determined for each family $(*)_{i \in I}$ of elements of $\mathcal{O}(R^{(I)})$.

It follows that $(f(i))_{i \in I}$ is a basis of $R^{(I)}$, by the Theorem 2.10. $\square$

Corollary 2.12. *Let R be a n -tuple ring and (M, g) a free R -module on a non-empty set I . Then $(g(i))_{i \in I}$ is a basis of M .*

Proof. Let $(R^{(I)}, f)$ be the free R -module on I , according to Theorem 2.7. By Theorem 2.6, there is a R -isomorphism $\varphi : R^{(I)} \rightarrow M$ such that $\varphi \circ f = g$. It follows that $\varphi(f(i)) = g(i)$, for all $i \in I$, which allows us to conclude that the family $(g(i))_{i \in I}$ is a basis of M , by Theorem 2.11. $\square$

Theorem 2.13. *Let R be a n -tuple ring and M a R -module. Then M is free if and only if it has a basis.*

Proof. From the hypothesis, there is a free R -module (F, f) on some non-empty I such that M is R -isomorphic to F . Since F has a basis, by Corollary 2.12, then M has a basis. Reciprocally, suppose that a family $(m_i)_{i \in I}$ of elements of M is a basis of M . Let $(R^{(I)}, f)$ be the free R -module on I , according to Theorem 2.7. Define $\iota : I \rightarrow M$ by $\iota(i) = m_i$ for all $i \in I$. Then, there is a unique R -homomorphism $h : R^{(I)} \rightarrow M$ such that $\iota = h \circ f$. Certainly, h is a R -epimorphism since $h(R^{(I)})$ is a submodule of M that contains the family $(m_i)_{i \in I}$. Now, if λ is an almost-null family of elements of R such that $h(\lambda) = 0$, then $\sum_{j \in I} \lambda_j *_{s_j} m_j = \sum_{j \in I} \lambda_j *_{s_j} \iota(j) =$

$\sum_{j \in I} \lambda_j *_{s_j} h(f(j)) = h \left(\sum_{j \in I} \lambda_j *_{s_j} f(j) \right) = h(\lambda) = 0$, according to the proof of the Theorem 2.7. It follows that $\lambda = 0$ which results h to be a R -monomorphism and therefore in a R -isomorphism. This show that M is free. $\square$

Corollary 2.14. *Let R be a n -tuple ring and M a free R -module. Then M is isomorphic to a direct sum of copies of R ; more precisely, if $(m_i)_{i \in I}$ is a basis of M then for each family $(*)_{s_i}^{i \in I}$ of elements of $\mathcal{O}(M)$ holds $M = \bigoplus_{i \in I} R *_{s_i} m_i$, where $R *_{s_i} m_i$ is R -isomorphic to R for each $i \in I$.*

Proposition 2.15. *Let R be a n -tuple ring, M a R -module and N a submodule of M . Then:*

- (i) *If M is finitely generated, then M/N is finitely generated;*
- (ii) *If N and M/N are finitely generated, then M is finitely generated;*
- (iii) *If N and M/N are free R -modules, then M is a free R -module.*

Proof. (i) It is obvious and therefore we will omit the proof here.

(ii) Consider $(m_i)_{i \in \{1, \dots, p\}}$ a finite family of elements of M such that the family $(m_i + N)_{i \in \{1, \dots, p\}}$ generates M/N and $(n_j)_{j \in \{1, \dots, q\}}$ a family of elements of N that generates N . For every element $m \in M$ let $(\lambda_i)_{i \in \{1, \dots, p\}}$ be a family of elements of R and $(*)_{s_i}^{i \in \{1, \dots, p\}}$ a family of elements of $\mathcal{O}(M)$ such that $m + N = \sum_{i=1}^p \lambda_i *_{s_i} (m_i + N)$. It follows that there is a family $(\mu_j)_{j \in \{1, \dots, q\}}$ of elements of R and a family $(*)_{r_j}^{j \in \{1, \dots, q\}}$ of elements of $\mathcal{O}(M)$ such that $m - (\sum_{i=1}^p \lambda_i *_{s_i} m_i) = \sum_{j=1}^q \mu_j *_{r_j} n_j$ which implies $m = \sum_{i=1}^p \lambda_i *_{s_i} m_i + \sum_{j=1}^q \mu_j *_{r_j} n_j$. This shows that M is finitely generated.

(iii) Consider $(m_i)_{i \in I}$ a family of elements of M such that $(m_i + N)_{i \in I}$ is a basis of M/N and $(n_j)_{j \in J}$ a basis of N . Define the family of elements of M $(u_k)_{k \in I \cup J}$ by

$$u_k = \begin{cases} m_k & , k \in I \\ n_k & , k \in J \end{cases}$$

Let's demonstrate that $(u_k)_{k \in I \cup J}$ is a basis of M . By hypothesis on N and M/N , for every element $m \in M$ and a family $(*)_{s_i}^{i \in I}$ of elements of $\mathcal{O}(M)$ there is a single almost-null family $(\lambda_i)_{i \in I}$ of elements of R such that $m + N = \sum_{i \in I} \lambda_i *_{s_i} (m_i + N)$ and for every family $(*)_{r_j}^{j \in J}$ of elements of

$\mathcal{O}(M)$ there is a single almost-null family $(\mu_j)_{j \in J}$ of elements of R such that $m - (\sum_{i \in I} \lambda_i \underset{s_i}{*} m_i) = \sum_{j \in J} \mu_j \underset{r_j}{*} n_j$. Define the families

$$(\nu_k)_{k \in I \cup J} = \begin{cases} \lambda_k & , k \in I \\ \mu_k & , k \in J \end{cases} \quad \text{and} \quad (\underset{t_k}{*})_{k \in I \cup J} = \begin{cases} \underset{s_k}{*} & , k \in I \\ \underset{r_k}{*} & , k \in J \end{cases} ,$$

then $m = \sum_{i \in I} \lambda_i \underset{s_i}{*} m_i + \sum_{j \in J} \mu_j \underset{r_j}{*} n_j = \sum_{k \in I \cup J} \nu_k \underset{t_k}{*} u_k$. This shows that $M = \langle (u_k)_{k \in I \cup J} \rangle$. Now, let $(\nu_k)_{k \in I \cup J}$ and $(\underset{t_k}{*})_{k \in I \cup J}$ be a almost-null family of elements of R and a family of elements of $\mathcal{O}(M)$, respectively, such that $\sum_{k \in I \cup J} \nu_k \underset{t_k}{*} u_k = 0$. It follows that $\sum_{k \in I} \nu_k \underset{t_k}{*} (m_k + N) = 0 + N$ which implies $\nu_k = 0$, for all $k \in I$, since the family $(m_k + N)_{k \in I}$ is a basis of M/N . Thus $\sum_{k \in J} \nu_k \underset{t_k}{*} n_k = 0$ which implies $\nu_k = 0$, for all $k \in J$, since the family $(n_k)_{k \in J}$ is a basis of N . Therefore, M is free. $\square$

Theorem 2.16. *Let R be a n -tuple ring and M, N R -modules such that M is free. If a family $(m_i)_{i \in I}$ is a basis of M , then every mapping $f : \{m_i\}_{i \in I} \longrightarrow N$ admits a unique extension to a R -homomorphism $\bar{f} : M \longrightarrow N$.*

Proof. Define the mapping $\bar{f} : M \longrightarrow N$ by

$$\bar{f} \left(\sum_{i \in I} \lambda_i \underset{r_i}{*} m_i \right) = \sum_{i \in I} \lambda_i \underset{r_i}{*'} f(m_i),$$

for all almost-null family $(\lambda_i)_{i \in I}$ of elements of R and every pair of family $(\underset{r_i}{*})_{i \in I}$ and $(\underset{r_i}{*'})_{i \in I}$ of elements of $\mathcal{O}(M)$ and $\mathcal{O}(N)$, respectively, such that the bilinear mappings $\underset{r_i}{*}$ and $\underset{r_i}{*'}$ correspond to the bilinear mapping r_i , for each $i \in I$. Note that this mapping is well-defined by virtue of the Theorem 2.10 and extends to a uniquely determined R -homomorphism. $\square$

Corollary 2.17. *Let R be a n -tuple ring and M a free R -module. If a family $(m_i)_{i \in I}$ is a basis of M , then M is isomorphic to $R^{(I)}$.*

Theorem 2.18. *Let R be a n -tuple ring and M a R -module. Then:*

(i) M is the image of a R -homomorphism of a free R -module;

(ii) M is R -isomorphic to quotient of a free R -module.

Proof. (i) Consider $(m_i)_{i \in I}$ a family of elements of M that generates M and $(f(i))_{i \in I}$ a basis of $R^{(I)}$, according to Theorem 2.11 and define the mapping $f : \{e_i\}_{i \in I} \rightarrow M$, by $f(e_i) = m_i$, for all $i \in I$. By Theorem 2.16, f extends to a uniquely determined R -epimorphism $\bar{f} : R^{(I)} \rightarrow M$.

(ii) The result is a direct consequence of $R^{(I)} / \ker(\bar{f}) \cong M$. $\square$

Proposition 2.19. *Let R be a n -tuple ring and L, M, N R -modules such that L is a free module. If $g : L \rightarrow N$ is a R -homomorphism and $f : M \rightarrow N$ is a R -epimorphism, then there is a R -homomorphism $\bar{g} : L \rightarrow M$ such that $f \circ \bar{g} = g$.*

Proof. Let $(l_i)_{i \in I}$ be a basis of L and we take the family $(g(l_i))_{i \in I}$, in N . From hypothesis on f , for every $i \in I$, there is an element $m_i \in M$ such that $f(m_i) = g(l_i)$. Define the mapping $g_1 : L \rightarrow M$, by $g_1(l_i) = m_i$, for all $i \in I$. Then g_1 extends to a uniquely determined R -homomorphism $\bar{g} : L \rightarrow M$, by Theorem 2.16. Thus, for every element $l \in L$, there is a family $(*)_{s_i}^{i \in I}$ of elements of $\mathcal{O}(L)$ and a single almost-null family $(\lambda_i)_{i \in I}$ of elements of R such that $l = \sum_{i \in I} \lambda_i *_{s_i} l_i$. Consider the families $(*)_{s_i}^{i \in I}$ and $(*)_{s_i}^{'' i \in I}$ of elements of $\mathcal{O}(M)$ and $\mathcal{O}(N)$, respectively, such that the bilinear mappings $*, *'_{s_i}$ and $''_{s_i}$ correspond to the bilinear mapping s_i , for each $i \in I$. Then, $f \left(\sum_{i \in I} \lambda_i *'_{s_i} m_i \right) = \sum_{i \in I} \lambda_i *''_{s_i} f(m_i)$ and $g \left(\sum_{i \in I} \lambda_i *_{s_i} l_i \right) = \sum_{i \in I} \lambda_i *''_{s_i} g(l_i)$. It follows that

$$\begin{aligned} (f \circ \bar{g})(l) &= f \left(\bar{g} \left(\sum_{i \in I} \lambda_i *_{s_i} l_i \right) \right) = f \left(\sum_{i \in I} \lambda_i *'_{s_i} \bar{g}(l_i) \right) \\ &= \sum_{i \in I} \lambda_i *''_{s_i} f(g_1(l_i)) = \sum_{i \in I} \lambda_i *''_{s_i} f(m_i) \\ &= \sum_{i \in I} \lambda_i *''_{s_i} g(l_i) = g \left(\sum_{i \in I} \lambda_i *_{s_i} l_i \right) = g(l). \end{aligned}$$

Therefore, $f \circ \bar{g} = g$. $\square$

3 Division rings

In this section, we obtain a characterization of the division rings, similar to classical theory. We prove that R is a division ring if and only if every left R -module on such rings is free. We begin with the following auxiliary lemma.

Lemma 3.1. *Let R a n -tuple ring. The following are equivalent:*

- (i) R is a division ring;
- (ii) Every nonzero element of R is s -left-invertible, for each $s = 1, \dots, n$;
- (iii) Every nonzero element of R is s -right-invertible, for each $s = 1, \dots, n$.

Proof. (i) $\implies$ (ii) and (i) $\implies$ (iii) are evident.

(ii) $\implies$ (i) Let a be a nonzero element of R . From the hypothesis, there exists an element $b_s \in R$ such that $b_s[s]a = 1_s$, for each $s = 1, \dots, n$. In particular, there is an element $c_s \in R$ such that $c_s[s]b_s = 1_s$, for each $s = 1, \dots, n$. It follows that $c_s = c_s[s](b_s[s]a) = (c_s[s]b_s)[s]a = a$, for each $s = 1, \dots, n$. Hence, a is also s -right-invertible, for each $s = 1, \dots, n$. This shows that R is a division ring.

(iii) $\implies$ (i) The proof is similar to the previous case (ii) $\implies$ (i). $\square$

Proposition 3.2. *Let R be a n -tuple ring. The following are equivalent:*

- (i) R is a division ring;
- (ii) The only left ideals of R are 0 and R ;
- (iii) The only right ideals of R are 0 and R .

Proof. (i) $\implies$ (ii) Let I be a nonzero left ideal of R and $a \in I$ a nonzero element. By Lemma 3.1, there are elements $b_s \in R$ such that $b_s[s]a = 1_s$, for each $s = 1, \dots, n$. Thus $1_s \in I$ for each $s = 1, \dots, n$. In particular, we obtain $I = R$.

(ii) $\implies$ (i) Let a be a nonzero element of R . For each $s = 1, \dots, n$, the additive subgroup $R[s]a = \{c[s]a \mid c \in R\}$ of R is a nonzero left ideal which results in $R[s]a = R$. It follows that the element a is s -left-invertible for each $s = 1, \dots, n$. Therefore, R is a division ring, by Lemma 3.1.

(i) $\implies$ (iii) and (iii) $\implies$ (i) are similarly proven. $\square$

Theorem 3.3. *Let R be a n -tuple ring. The following are equivalent:*

- (i) R is a division ring;
- (ii) Every R -module M is free.

Proof. (i) $\implies$ (ii) Let M be a R -module with more than one element. Consider Δ the set of all linearly independent families of M , partially ordered by set inclusion. Certainly, Δ is nonempty and since every totally ordered subset of Δ has an upper bound, then Δ contains a maximal element $(m_i)_{i \in I}$, by Zorn's Lemma. If $M \neq \langle (m_i)_{i \in I} \rangle$, then there is an element $m_{i_0} \in M$ with $m_{i_0} \notin \langle (m_i)_{i \in I} \rangle$. Consider the family $(m_i)_{i \in I \cup \{i_0\}}$ and let's demonstrate that it is linearly independent. Let $(\lambda_i)_{i \in I \cup \{i_0\}}$ be an almost-null family of elements of R and $(*)_{i \in I \cup \{i_0\}}$ be a family of elements of $\mathcal{O}(M)$ such that $\lambda_{i_0} *_{s_{i_0}} m_{i_0} + \sum_{i \in I} \lambda_i *_{s_i} m_i = 0$. If $\lambda_{i_0} \neq 0$, then there is an element $\mu_{i_0} \in R$ such that $\mu_{i_0} \boxed{s_{i_0}} \lambda_{i_0} = 1_{s_{i_0}}$ which implies that $m_{i_0} = \sum_{i \in I} (-\mu_{i_0} \boxed{s_{i_0}} \lambda_i) *_{s_i} m_i$ which contradicts the choice of the element m_{i_0} . Hence $\lambda_{i_0} = 0$ which implies $\lambda_i = 0$, for all $i \in I$, since $(m_i)_{i \in I}$ is a linearly independent family. Consequently $(m_i)_{i \in I \cup \{i_0\}}$ is a linearly independent family of M , contradicting the maximality of $\{m_i\}_{i \in I}$. Therefore $M = \langle (m_i)_{i \in I} \rangle$ and $(m_i)_{i \in I}$ is a basis of M . (ii) $\implies$ (i) In particular, consider M a simple R -module. By Corollary 2.14, M is isomorphic to a direct sum of copies of R and therefore isomorphic to R . It follows that R itself is simple as a R -module and so the only nonzero left ideal of R is R itself. Thus, R is a division ring, by Proposition 3.2(ii). $\square$

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ACCURATE MASS FORMULAS AND PREDICTION
OF MASSES ON HEAVY FLAVOR HADRONS

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Abstract

Based on the emergence string and other mechanism, we may derive the GMO mass formula and its modified accurately form $M = M_0 + AS + B[I(I + 1) - S^2 / 2]$.

According to the symmetry of s-c quarks, the heavy flavor hadrons which made of u, d and c quarks may be classified by SU(3) octet and decuplet, and some simple mass formulas are obtained, then $m(\Xi_{cc})=3715$ or $3673MeV$ are predicted. In July

2017 LHC observed a new and very charming baryon $\Xi_{cc}^{++} = ucc$, whose mass is $3621MeV$. It agrees more on new formula, whose error only is 1.4%. Further, we predict $m(\Omega_{cc}^+)=3946$, or 3950.8 , or $3908.2MeV$, etc. Moreover, this method may

be extended to other heavy flavor hadrons, and predict $m(\Xi_{bb})=10396.8$ or $10348.9MeV$, etc. It is a quantitative and testable theory.

Key words: hadron, mass formula, string, heavy flavor.

1. Introduction

It is very successful that the SU(3) symmetry and its broken are applied to the classification of the ground hadrons, which made of u, d and s quarks, and to the mass spectrum, etc. It is also well known that those hadrons agree with the Gell-Mann-Okubo (GMO) mass formula:

$$M = M_0 + AS + B[I(I+1) - \frac{S^2}{4}]. \quad (1)$$

Here must assume that $M = m^2$ for mesons. Recently, Koschmieder discussed the status of particle theory, and investigated the masses of the elementary particles [1]. In this paper, we discuss the modified accurate mass formulas, and from which predict the masses of some heavy flavor hadrons.

Recently, Gorbunov and Tokareva shown a framework of high energy scale invariance can explain the origin of dark radiation [8]. Pérez-García and Silk discussed constraining decaying dark matter with neutron stars [9]. Miranda, et al.,

2. Emergence String and Accurate Mass Formulas of Hadrons

By different ways [2-5], we derived the GMO mass formula (1) and its modified accurate mass formula:

$$M = M_0 + AS + B[I(I+1) - \frac{S^2}{2}]. \quad (2)$$

Simultaneously, we obtained the corresponding symmetric lifetime formulas of hadrons [3-5].

Using Eq.(2), for the $J^P = 1^+ / 2$ baryon octet, let $M_0 = 910.75 MeV$, $A = -222.04 MeV$ and $B = 38.43 MeV$, so

$$\begin{aligned} m(n) &= 939.5725, m(\Lambda) = 1115.6 MeV, \\ m(\Sigma^0) &= 1192.46, m(\Xi^0) = 1314.89 MeV. \end{aligned} \quad (3)$$

Therefore, Eq.(2) agrees completely with the experimental data of the neutral baryons [6].

For the $J^{PC} = 0^{-+}$ meson octet, let $A = 0$, $M_0 = 549.4 MeV$ and $B = -207.22 MeV$, so

$$m(\pi^0) = 134.96, m(K^0) = 497.6, m(\eta) = 549.4 \text{ MeV}, \quad (4)$$

the neutral mesons agree completely within the range of error [6], and baryon and meson can be unified by $M=m$.

For the $J^P = 3^+ / 2$ baryon decuplet $I=1+(B+S)/2$ always holds, so the formula (2) just derives a more accurate formula

$$M = (M_0 + \frac{15}{4}B) - (A - 2B)S - \frac{B}{4}S^2 = M_0' - aS + bS^2. \quad (5)$$

Let $M_0' = 1231.8$, $a = 156.8$ and $b = -3.3 \text{ MeV}$, so

$$m(\Delta) = 1231.8, m(\Sigma) = 1385.3, m(\Xi) = 1532.2, m(\Omega) = 1672.5 \text{ MeV}. \quad (6)$$

The mass relations

$$2(N + \Xi) = 3\Lambda + \Sigma, \quad (7)$$

$$4(N + \Xi) = 7\Lambda + \Sigma, \quad (8)$$

correspond to Eqs.(1) and (2), respectively.

Based on the known theories [7,8], the mass operator of the first broken SU(3) symmetry is [7]:

$$O_M = H_{VSI} + H_{MSI} = M_0 + AU_3 = M + AS + BI(I+1) + CS^2. \quad (9)$$

For the famous mass relation (1), $C = -B/4$ is obtained. For a new mass relation (2) which agrees better with the data, we may obtain $C = -B/2$ and Eq.(2). In Ref.[8], the mass of the same multiplet can be split into irreducible parts according to the Clebsch-Gordan series

$$8 \otimes 8 = 1 \oplus 8' \oplus 8'' \oplus 10 \oplus \overline{10} \oplus 27. \quad (10)$$

Neutrality means that 10 and $\overline{10}$ are absent, then

$$m_N = (m_1 - 2m'_8 + m''_8 - 3m_{27}), m_\Lambda = (m_1 - m'_8 - m''_8 + 9m_{27}), \quad (11)$$

$$m_\Sigma = (m_1 + m'_8 + m''_8 + m_{27}), m_\Xi = (m_1 + m'_8 - 2m''_8 - 3m_{27}). \quad (12)$$

“Suppose that the mass splitting is generated in a more fundamental Lagrangian by a

quark-mass term of the type $\bar{\psi}(a + b\lambda_8)\psi$ carrying only singlet and octet representations and that this property is (miraculously) preserved by the interactions. This would mean that m_{27} vanishes," leading to Eq.(1). If we let $m_{27} = (m'_8 + m''_8) / 44 \neq 0$, Eq.(2) will be derived.

Further, we research possible mechanism of formulas (1) and (2), which include that $I(I+1)$ corresponds to rotation, and S and S^2 corresponds to oscillation, in particular, for anharmonicity. Assume that hadrons are formed from the emergence string [5]. Usual string should possess two moving states: oscillation and rotation, based on the general oscillation and rotation of string we propose that a corresponding equation of the emergence string may be

$$\frac{d^2\psi}{dr^2} + \left[-\frac{K(K+1)}{r^2} + 2m(E-U)\right]\psi = 0. \quad (13)$$

This equation may regard one of a scalar field in the gauge theory derived from emergence string [9]. Moreover, if U is a potential of the anharmonic oscillator [10,11]:

$$U(r) = \frac{1}{2}m\omega^2 r^2 + \alpha r^3. \quad (14)$$

Bolokhov, et al., described the heterotic string solutions in the M model, which obtained in $U(N)$ gauge theories for $N=2$ supersymmetric QCD deformed by superpotential terms μA^2 breaking [12]. Polyakov constructed vertex operators for massless higher spin fields in Ramond-Neveu-Schwarz superstring theory, and derived the gauge-invariant cubic interaction terms for the higher spins [13].

Such a corresponding energy level for Eq.(13), in second approximation, is:

$$E_{k,n} = E_{0,0} + \hbar\omega\left(n + \frac{1}{2}\right) + \frac{\hbar^2}{2mr_0^2}K(K+1) - \frac{\hbar^2\alpha^2}{8m^3\omega^4}(30n^2 + 30n + 11) \quad .$$

(15)

If oscillation and rotation of the emergence string correspond respectively to quantum numbers $S(Y)$ and I , this energy level will correspond completely to the mass formula of hadrons:

$$M = M_0 + AS + BI(I + 1) + CS^2. \quad (16)$$

When $C = -B/4$, it is the GMO mass formula (1). When $C = -B/2$, it is a modified accurate mass formula (2)[2-5]. Such not only baryons and mesons are unified $M=m$, and Eq.(2) agrees accurately with the experiments. These are some relations between the string and observable experimental data.

Further, any model with general oscillation and rotation may obtain the Morse potential [14] and corresponding energy level.

3. Heavy Flavor Hadrons and Their Masses

In the standard model quarks are the three generations (u,d) (c,s) and (t,b), whose properties exhibit a better symmetry. Some hadrons including heavy flavor c, b and t quarks have been found, and they are consistent with the SU(N) multiplets. Based on the symmetry of s and c quarks in the same generation, we supposed that the hadrons, which made of u, d and c quarks, are also the SU(3) symmetry and are classified by octet and decuplet [15,4,5]. It is a subgroup of SU(4) of u, d, s and c quarks. Such the eight $J^P = 1^+ / 2$ baryons form also an octet:

$$p=uud, n=udd(I=1/2);$$

$$\Lambda_c^+ = udc(I = 0); \Sigma_c^{++} = uuc, \Sigma_c^+ = udc, \Sigma_c^0 = ddc(I = 1);$$

$$\text{and } \Xi_{cc}^{++} = ucc, \Xi_{cc}^+ = dcc(I = 1/2).$$

Such we assume that these masses of the octet obey the corresponding simple mass formulas only by $S \rightarrow C$ in Eqs.(1) and (2):

$$M = M_0 + AC + B[I(I + 1) - (C^2 / 4)], \quad (17)$$

$$\text{or } M = M_0 + AC + B[I(I + 1) - (C^2 / 2)]. \quad (18)$$

Since $m(N)=939$, $m(\Lambda_c^+) = 2285, m(\Sigma_c) = 2453 \text{ MeV}$ [6], from the two corresponding mass formulas (7) and (8) we predicted $m(\Xi_{cc}) = 3715$ or 3673 MeV [15,4,5]. In 10 July 2017 LHC announced to observe a new and very charming baryon

$\Xi_{cc}^{++} = ucc$, whose mass is $3621.40 \pm 0.72(\text{stat.}) \pm 0.27(\text{syst.}) \text{MeV}$, and decay mode is $\Xi_{cc}^{++} \rightarrow \Lambda_c^+ K^- \pi^+ \pi^+$ [16]. New experimental data agree more on Eq.(8) and (2), whose error only is $(3673-3621)/3621=1.4\%$. Moreover, there should have $\Xi_{cc}^+ = dcc$, both form $I=1/2$ doublet with near mass, and one of decay modes is $\Xi_{cc}^+ \rightarrow \Lambda_c^+ K^- \pi^+ \pi^0$ [17]. In 1994 we predicted that “if the experiments derive this mass, it will show that our theory (hadrons made of u, d and c quarks are the SU(3) symmetry) is right”[15].

According new experimental data [6], let $M_0=877.2$, $A=1450.9$, $B=83.2 \text{MeV}$ in (18), we derive $m(n)=939.6$, $m(\Lambda_c^+) = 2286.5$, $m(\Sigma_c^+) = 2452.9 \text{ MeV}$ and $m(\Xi_{cc}) = 3675 \text{MeV}$.

For the $J^P = 3^+ / 2^-$ baryons, $\Delta^{++}, \Delta^+, \Delta^0, \Delta^-$ ($I=3/2$); $\Sigma_c^{++}, \Sigma_c^+, \Sigma_c^0$ ($I=1$); $\Xi_{cc}^{++}, \Xi_{cc}^+$ ($I=1/2$) and $\Omega_{ccc}^{++} = ccc$ ($I=0$) form also a decuplet. Their masses are possibly an equal-spacing rule, i.e.

$$M = M_0 + aC. \quad (19)$$

Of course, these baryons obey probably the more accurate mass formulas

$$M = M_0 + aC + bC^2, M = M_0 + a'I + b'I^2, \quad (20)$$

which correspond to Eqs.(2) and (5).

The $J^P = 0^-$ octet of heavy flavor mesons are:

$\pi^{+,0,-}, \pi^0$ ($I = 1$); $D^+ = c\bar{d}, D^0 = c\bar{u}$ ($I = 1/2$) and their antiparticles;

$\eta'_c = a(u\bar{u} + d\bar{d}) + b(c\bar{c})$. If their mass relation is:

$$4m(D) = m(\pi) + 3m(\eta'_c) \text{ or } 8m(D) = m(\pi) + 7m(\eta'_c), \quad (21)$$

so $m(\eta'_c) = 2444$ or $2114 MeV$ since $m(\pi) = 137$, $m(D) = 1867 MeV$. For the $J^P = 1^-$ octet, $m(\rho) = 770$, $m(D^+) = 2010 MeV$, so $m(\eta'_c) = 2423$ or $2187 MeV$.

For the $J^P = 1^+ / 2$ baryons,

$$m(\Xi_c^+(usc)) = 2467.8 MeV, m(\Xi_c^0(dsc)) = 2470.9 MeV (I=1/2),$$

$$m(\Omega_c(ssc)) = 2695.2 MeV.$$

These octets and decuplet are a certain cross section of the diagrams of the SU(4) multiplets, respectively. For the $J^P = 3^+ / 2$ baryons, probably, the masses of the triplet $\Sigma^+, \Sigma^0, \Sigma^-$ (I=1), the doublet $\Xi_c^+(usc), \Xi_c^0(dsc)$ (I=1/2), and the singlet

$\Omega_{cc}^+ = scc$ (I=0) are approximately an equal-spacing rule, it is the second series of series

of heavy flavor baryons. Then the masses of $\Omega^- = sss$, $\Omega_c^0 = ssc$, Ω_{cc}^+ and Ω_{ccc}^{++} are only four hadrons for mixtures of second generation, and their masses are should be equal-spacing too. It is the third series of heavy flavor baryons. For the $J^P = 3^+ / 2$ baryons, $m(\Sigma_c) = 2518$, $m(\Xi_c) = 2646$, $m(\Omega_c(ssc)) = 2766 MeV$, these masses obey equal-spacing rule. Such based on the known masses of $3^+ / 2$ baryons including c quark [6], other masses of baryons will be able to be estimated.

Because $\Sigma_c(uuc, 2518) - \Delta(uuu, 1232) = 1286 \approx m(c) - m(u)$, so $m(\Xi_{cc}) = 3804$ and $m(\Omega_{ccc}^{++}) = 5090$. From the third series, we obtain $m(\Omega^-) = 1675.4$, $m(\Omega_c^0) = 2811.6$, $m(\Omega_{cc}^+) = 3950.8$ and $m(\Omega_{ccc}^{++}) = 5090$. But, from the second series the known $m(\Sigma) = 1385$ and $m(\Xi_c) = 2646.6$, so $c - u \approx 1261.6$ and $m(\Omega_{cc}^+) = 3908.2$. Both $m(\Omega_{cc}^+)$ are different due to different of spin. Of course, these baryons obey probably the more accurate mass formulas (20).

Such any one of masses of $3^+ / 2$ baryons including c quark is known again, for example, for $\Sigma_c^0(2517.5 \pm 1.4) MeV$ ($J^P = 3^+ / 2$) [18], then other five masses of other baryons will be able to be estimated.

baryon	Ω^-	Ξ_c^-	Δ^-	Σ_c^-	Ξ_{cc}^-	Ω_c^-	Ω_{cc}^+	Ω_{ccc}^{+++}
$m(\text{exp})$	1672	2644	1231	2516				
$m(\text{est})$	input	2666	input	input	3800	2809	3946	5084

Therefore, by using different ways we predict different $m(\Omega_{cc}^+) = 3946$, or 3950.8, or 3908.2 MeV, etc.

For the doublet $\Xi_c^+ = usc, \Xi_c^0 = dsc$ ($I=1/2$, $S=-1$, $C=1$) and both masses are 2467.9 and 2471.0 MeV, the singlet $\Omega_c^0 = ssc$ ($I=0$, $S=-2$, $C=1$) and its mass is 2697.5 $\pm$ 2.6 MeV, and the singlet $\Omega_{cc}^+ = scc$ ($I=0$, $S=-1$, $C=2$), we can assume that the simplest mass formula is:

$$M = M_0 + A(C - S) + B[I(I + 1) - (C - S)^2 / 2], \quad (22)$$

$$\text{or} \quad M = M_0 + A(C - S) + B[I(I + 1) - (C^2 + S^2) / 2]. \quad (23)$$

Some similar multiplets and mass formulas exist possibly in baryons and mesons including b or t quarks. For instance, both mass spectra of $\psi = c\bar{c}$ and $\gamma = b\bar{b}$ are similar; as in the neutral kaon system, $D^0 - \bar{D}^0$ and $B^0 - \bar{B}^0$ mixings should exist.

Further, this method will be able to be extended to other potentials and other heavy flavor hadrons. For example, based on the symmetry of quarks model, we may suppose that the hadrons, which made of u, d and b quarks, and of u, d and t all are the SU(3) symmetry. It is a subgroup of SU(4) of u, d, b and t quarks. Such the eight $J^P = 1^+ / 2$ baryons form also an octet:

$$p=uud, n=udd(I=1/2); \Lambda_b^0 = udb(I=0);$$

$$\Sigma_b^+ = uub, \Sigma_b^0 = udb, \Sigma_b^- = ddb(I=1);$$

$$\text{and } \Xi_b^0 = ubb, \Xi_b^- = dbb(I=1/2).$$

Such the corresponding mass formulas are:

$$M = M_0 + aB + b[I(I+1) - (B^2/4)], \quad (24)$$

$$\text{or } M = M_0 + aB + b[I(I+1) - (B^2/2)]. \quad (25)$$

It is known the $m(\Lambda_b)=5620$, $m(\Sigma_b)=5811.5(5807.8, 5815.2)MeV$ [6]. From the two corresponding mass formulas, we may predict $m(\Xi_{bb})=10396.8$ or $10348.9MeV$, whose error is probably bigger due to weaker symmetry between first (u,d) and third generations (t,b).

Generally, based on the symmetry of quark model we can suppose that the hadrons, which made of s, c and b quarks are the SU(3) symmetry. Similar, the hadrons made of s, c and t quark are also the SU(3) symmetry. Both are two subgroups of SU(4) of s, c, b and t quarks, but these quarks are all $I=0$ and very unstable. These mass formulas are possibly:

$$M = M_0 + aS + a'C + a''B + bS^2 + b'C^2 + b''B^2, \quad (26)$$

$$\text{or } M = M_0 + aS + a'C + a''B + b(S + C + B)^2. \quad (27)$$

The mixtures of three generations are:

$$m(\Xi_b^0 = usb, \Xi_b^- = dsb)=5790.5MeV(I=1/2).$$

$$\text{Other hadrons are } m(\Lambda_b^0(udb))=5620.2MeV, m(\Omega_b^-(ssb))=6071MeV, \text{ etc.}$$

In a word, our research based some symmetries among different generations is a quantitative and testable theory.

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THE LIE SANTILLI'S WEAK-HYPER-ADMISSIBILITY.
THE HELIX-HYPEROPERATIONS ON THE
LOW DIMENSIONAL CASES.

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Abstract

A very important new field in hypermathematics comes straightforward from *Santilli's Admissibility* introduced in 1968. We can transfer Santilli's theory in admissibility for representations in the general case, in two ways: using either, the ordinary matrices and a hyperoperation on them, or using hypermatrices and ordinary operations on them. In this paper we focus on the admissibility by using non-square matrices using the, so called, *helix-hyperoperations* which are hyperoperations defined on every type of ordinary matrices. New types of matrices are introduced which with the helix-hyperoperations can face the novel theories on Lie-Santilli's admissibility

Key words: *hyperstructure, H_v-structure, hope, Lie-Santilli's admissibility.*

AMS Subject Classification: 20N20, 16Y99

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1. INTRODUCTION

We deal with the largest class of hyperstructures called *H_v-structures* introduced by Vougiouklis in 1990 [14], [15], which satisfy the *weak axioms* where the non-empty intersection replaces the equality.

Definitions 1.1 In a set H with a *hyperoperation* (*=hope*) $\cdot: H \times H \rightarrow P(H)$, we abbreviate by

WASS the *weak associativity*: $(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H$ and by

COW the *weak commutativity*: $xy \cap yx \neq \emptyset, \forall x, y \in H$.

The hyperstructure $(H, \cdot)$ is called *H_v-semigroup* if it is WASS and it is called *H_v-group* if it is reproductive *H_v-semigroup*: $xH = Hx = H, \forall x \in H$.

A set R equipped with two hopes $(+)$ and $(\cdot)$, $(R, +, \cdot)$ is called *H_v-ring* if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \emptyset, (x+y)z \cap (xz+yz) \neq \emptyset, \forall x, y, z \in R.$$

An extreme class is: An *H_v-structure* is *very thin* iff all hopes are operations except one, with all hyperproducts singletons except one, which is subset of cardinality more than one.

The *fundamental relations* β^* and γ^* are defined, in *H_v-groups* and *H_v-rings*, respectively, as the smallest equivalences so that the quotient would be group and ring, respectively. The main theorem for the fundamental relation β^* is the following:

Theorem 1.2 Let $(H, \cdot)$ be *H_v-group* and denote U the set of finite products of elements of H . Define a relation β in H as follows: $x\beta y$ iff $\{x, y\} \subset u$ where $u \in U$. Then the fundamental relation β^* is the transitive closure of β .

Analogous theorem for the fundamental relation γ^* in *H_v-rings*.

An element is called *single* if its fundamental class is a singleton.

Since 1990 a lot of *H_v-structures* appeared, new classes and applications studied. One can see few of them in [1], [2], [3], [11], [12], [15], [16], [17], [18], [19], [22], [27], [28], [30], [31], [32], [34].

Definition 1.3 Let $(H, \cdot)$, $(H, \otimes)$ be *H_v-semigroups* defined on the same H . $(\cdot)$ is *smaller* than $(\otimes)$, and $(\otimes)$ *greater* than $(\cdot)$, iff there is automorphism

$f \in \text{Aut}(H, \otimes)$ such that $xy \subset f(x \otimes y)$, $\forall x \in H$.

$(H, \otimes)$ contains $(H, \cdot)$. If $(H, \cdot)$ is structure, then $(H, \otimes)$ is an ***H_b-structure***.

The Little Theorem. Greater hopes of the ones which are WASS or COW, are also WASS and COW, respectively.

This theorem indicates that we can construct a great number of hopes and hyperstructures from an operation or a hope.

The fundamental relations help to researchers to give definitions which were not possible without them. Therefore, there was not any general definition for a hyperfield but now there is the following.

Definition 1.4 The H_v -ring $(R, +, \cdot)$ is ***H_v-field*** if the quotient R/γ^* is a field.

Moreover, new hyperstructures appear:

Definition 1.5 The H_v -semigroup $(H, \cdot)$ is ***h/v-group*** if H/β^* is a group.

Definition 1.6 Let $(R, +, \cdot)$ be H_v -ring, $(M, +)$ COW H_v -group and external hope $\cdot: R \times M \rightarrow P(M): (a, x) \rightarrow ax$, such that, $\forall a, b \in R$ and $\forall x, y \in M$ we have

$$a(x+y) \cap (ax+ay) \neq \emptyset, (a+b)x \cap (ax+bx) \neq \emptyset, (ab)x \cap a(bx) \neq \emptyset,$$

then M is called an ***H_v-module*** over R . In the case of an H_v -field F instead of H_v -ring R , then the ***H_v-vector space*** is defined.

The most strict and useful in physics, hyperstructure is the Lie Algebra which is defined in hyperstructures, mainly in [22], and then for several applications as follows [13], [15], [19], [21], [25], [27], [30]:

Definition 1.7 Let $(L, +)$ H_v -vector space on $(F, +, \cdot)$, $\varphi: F \rightarrow F/\gamma^*$, canonical map and $\omega_F = \{x \in F: \varphi(x) = 0\}$, where 0 is the zero of the fundamental field F/γ^* . Similarly, let ω_L be the core of the canonical map $\varphi': L \rightarrow L/\varepsilon^*$ and denote again 0 the zero of L/ε^* . Consider the ***bracket*** (commutator) hope:

$$[,] : L \times L \rightarrow P(L): (x, y) \rightarrow [x, y]$$

then L is an ***H_v-Lie algebra*** over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \emptyset$$

$$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \emptyset, \quad \forall x, x_1, x_2, y, y_1, y_2 \in L, \lambda_1, \lambda_2 \in F$$

$$(L2) \quad [x, x] \cap \omega_L \neq \emptyset, \quad \forall x \in L$$

$$(L3) \quad ([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \emptyset, \quad \forall x, y \in L$$

A large class of H_V -structures, based on classical, is the following:

Definition 1.8 Let $(G, \cdot)$ groupoid, then $\forall P \subset G, P \neq \emptyset$, the ***P-hopes*** are:

$$\underline{P}: x \underline{P} y = (xP)y \cup x(Py),$$

$$\underline{P}_r: x \underline{P}_r y = (xy)P \cup x(yP), \quad \underline{P}_l: x \underline{P}_l y = (Px)y \cup P(xy), \quad \forall x, y \in G$$

$(G, \underline{P})$, $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ are called ***P-hyperstructures***. If $(G, \cdot)$ is semigroup, then $x \underline{P} y = (xP)y \cup x(Py) = xPy$ and $(G, \underline{P})$ is a semihypergroup

A generalization of *P-hopes*, used in Santilli's isothory, is the following: Let $(G, \cdot)$ abelian group and P with $\#P > 1$. Define the hope $(\times_P)$ as follows:

$$x \times_P y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y \mid h \in P\} & \text{if } x \neq e \text{ and } y \neq e \\ x \cdot y & \text{if } x = e \text{ or } y = e \end{cases}$$

we call this hope P_e -hope. The hyperstructure $(G, \times_P)$ is abelian H_V -group.

Last years, hyperstructures have a variety of applications in mathematics and other sciences. The hyperstructures theory is now widely applicable in industry and production, too. The theory of matrix representations is very important topic in which several hyperstructures are used.

2. THE ISO-HYPERNUMBERS

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems [4], [5], [7]. Santilli proposed a 'lifting' of the n -dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n -dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond to the hyperstructures introduced by Santilli & Vougiouklis in 1996 [8], and they are called *e-hyperfields*. For a presentation and results on iso-hyper-theory, see [1], [2], [6], [8], [20], [25], [26], [29].

Definition 2.1 A hyperstructure $(H, \cdot)$ containing a unique scalar unit e and $\forall x$, there exists an inverse x^{-1} , i.e. $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$, is an *e-hyperstructure*.

Definition 2.2 $(F, +, \cdot)$, $(+)$ operation $(\cdot)$ hope, is *e-hyperfield* if: $(F, +)$ abelian group with 0; $(\cdot)$ is WASS; $(\cdot)$ weak distributive to $(+)$; 0 absorbing: $0 \cdot x = x \cdot 0 = 0$, $\forall x \in F$; there is scalar unit 1: $1 \cdot x = x \cdot 1 = x$, $\forall x \in F$, and $\forall x \in F$ there is a unique inverse x^{-1} : $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$. The elements of an e-hyperfield are called *e-hypernumbers*. If $1 = x \cdot x^{-1} = x^{-1} \cdot x$, then we have a *strong e-hyperfield*.

Definition 2.3 Main e-Construction. Let $(G, \cdot)$, be a group, e the unit, then we can define in G , great number of hopes $(\otimes)$ as follows:

$$x \otimes y = \{xy, g_1, g_2, \dots\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, \dots \in G - \{e\}$$

where $g_1, g_2, \dots$ are not the same for all $x, y \in G - \{e\}$. The hyperstructure $(G, \otimes)$ is an H_b -group which contains the $(G, \cdot)$ so it is an H_b -group. Moreover $(G, \otimes)$ is an e-hypergroup. If $\forall x, y$ such that $x \otimes y = xy = e$, then $(G, \otimes)$ is a strong e-hypergroup.

In Santilli's iso-theory, on a field $F = (F, +, \cdot)$, a *general isofield* $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ is defined to be a field with elements $\hat{a} = a \times \hat{1}$, called *isonumbers*, where $a \in F$, and $\hat{1}$ is a positive-defined element generally outside F , equipped with two operations $\hat{+}$ and $\hat{\times}$ where $\hat{+}$ is the sum with the conventional additive unit 0, and $\hat{\times}$ is a new multiplication

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \times \hat{T} \times \hat{b}, \quad \text{with } \hat{1} = \hat{T}^{-1}, \quad \forall \hat{a}, \hat{b} \in \hat{F} \quad (i)$$

called *iso-multiplication*, for which $\hat{1}$ is the left and right unit of $\hat{F}$, i.e.

$$\hat{1} \hat{\times} \hat{a} = \hat{a} \times \hat{1} = \hat{a}, \quad \forall \hat{a} \in \hat{F} \quad (ii)$$

called *iso-unit*. The rest properties of a field are reformulated analogously.

Construction 2.4 On field $F = (F, +, \cdot)$ and isofield $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ we replace in the results of the iso-product

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \times \hat{T} \times \hat{b}, \quad \text{with } \hat{1} = \hat{T}^{-1}$$

of $\hat{T}$ by a set containing $\hat{T}$, $\hat{H}_{ab} = \{\hat{T}, \hat{x}_1, \hat{x}_2, \dots\}$, where $\hat{x}_1, \hat{x}_2, \dots \in \hat{F}$, for all hyperproducts $\hat{a} \hat{\times} \hat{b}$ for which $\hat{a}, \hat{b} \notin \{\hat{0}, \hat{1}\}$ and $\hat{x}_1, \hat{x}_2, \dots \in \hat{F} - \{\hat{0}, \hat{1}\}$. If one of $\hat{a}$, $\hat{b}$, or both, is equal to $\hat{0}$ or $\hat{1}$, then $\hat{H}_{ab} = \{\hat{T}\}$. Thus, the new iso-hope is

$$\hat{a} \times \hat{b} = \hat{a} \times \hat{H}_{ab} \times \hat{b} = \hat{a} \times \{\hat{T}, \hat{x}_1, \hat{x}_2, \dots\} \times \hat{b}, \forall \hat{a}, \hat{b} \in \hat{F} \quad (\text{iii})$$

$\hat{F} = \hat{F}(\hat{a}, \hat{+}, \times)$ is *isoH_v-field*, which elements are *isoH_v-numbers*.

Important hopes are if only for few pairs $(\hat{a}, \hat{b})$ the result is enlarged. Even more in the special case is if there exists only one pair $(\hat{a}, \hat{b})$ for which

$$\hat{a} \times \hat{b} = \hat{a} \times \{\hat{T}, \hat{x}\} \times \hat{b}, \forall \hat{a}, \hat{b} \in \hat{F}$$

then we obtain the *very thin isoH_v-field*.

Construction 2.5 Consider isofield $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \times)$ with $\hat{a} = a \times \hat{1}$, isonumbers, where $a \in F$, and $\hat{1}$ is a positive-defined element generally outside F , with two operations $\hat{+}$ and $\times$, where $\hat{+}$ is the sum with the conventional additive unit 0, and $\times$ is the iso-multiplication

$$\hat{a} \times \hat{b} := \hat{a} \times \hat{T} \times \hat{b}, \quad \text{with } \hat{1} = \hat{T}^{-1}, \forall \hat{a}, \hat{b} \in \hat{F}$$

Take $\hat{P} = \{\hat{T}, \hat{p}_1, \dots, \hat{p}_s\}$, with $\hat{p}_1, \dots, \hat{p}_s \in \hat{F} - \{\hat{0}, \hat{1}\}$, we define the *iso-P-H_v-field*, $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \times_P)$ where the hope $\times_P$ is defined as follows:

$$\hat{a} \times_P \hat{b} := \begin{cases} \hat{a} \times \hat{P} \wedge \hat{b} = \{\hat{a} \times \hat{h} \wedge \hat{b} \mid \hat{h} \in \hat{P} \wedge\} & \text{if } \hat{a} \neq \hat{1} \text{ and } \hat{b} \neq \hat{1} \\ \hat{a} \times \hat{T} \wedge \hat{b} & \text{if } \hat{a} = \hat{1} \text{ or } \hat{b} = \hat{1} \end{cases} \quad (\text{iv})$$

The elements of $\hat{F}$ are *iso-P-H_v-numbers*. The inverses are not unique.

Remarkable is when $\hat{P} = \{\hat{T}, \hat{p}\}$, that is that $\hat{P}$ contains only one $\hat{p}$ except $\hat{T}$.

Non degenerate examples on the above constructions on the small finite field $(Z_7, +, \cdot)$. In the $\hat{Z}_7 = \hat{Z}_7(\hat{a}, \hat{+}, \times)$, where we denote $\hat{Z}_7 = \{\hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{4}, \hat{5}, \hat{6}\}$, the WASS multiplicative hope is described by only one enlargement $\hat{2} \times \hat{6} = \{\hat{2}, \hat{5}\}$

Some hopes 'like' P-hopes are the following:

Definition 2.6 Let $M = M_{m \times n}$ be the $m \times n$ matrices over R and $P = \{P_i; i \in I\} \subseteq M$. We define, a kind of, a P-hope $\underline{P}$ on M as follows

$$\underline{P}: M \times M \rightarrow P(M): (A, B) \underline{A} \underline{P} B = \{A P_i^t B; i \in I\} \subseteq M$$

where P^t denotes the transpose of P .

The hope $\underline{P}$ is a bilinear map, strong associative and inclusion distributive:

$$\underline{A} \underline{P} (B + C) \subseteq \underline{A} \underline{P} B + \underline{A} \underline{P} C, \forall A, B, C \in M$$

Thus, $(M, +, \underline{P})$ defines a multiplicative hyperring on non-square matrices.

Let $M = M_{m \times n}$ be a module of $m \times n$ matrices over R and let us take sets

$$S = \{s_k : k \in K\} \subseteq R, \quad Q = \{Q_i : i \in J\} \subseteq M, \quad P = \{P_i : i \in I\} \subseteq M.$$

Define three hopes as follows

$$\underline{S} : R \times M \rightarrow P(M) : (r, A) \rightarrow r \underline{S} A = \{(rs_k)A : k \in K\} \subseteq M$$

$$Q_+ : M \times M \rightarrow P(M) : (A, B) \rightarrow A Q_+ B = \{A + Q_j + B : j \in J\} \subseteq M$$

$$P : M \times M \rightarrow P(M) : (A, B) \rightarrow A P B = \{A P_i B : i \in I\} \subseteq M$$

$(M, \underline{S}, Q_+, P)$ is hyperalgebra on R called *general matrix P-hyperalgebra*.

3. THE HELIX-HOPEs

Since the sum and the product of matrices are partial operations one can see that there are several cases to face the problem using hyperstructures. One very interesting case, where all entries of matrices are used, introduced in [31], and it is defined as follows [3], [11], [12], [32], [34]:

Definition 3.1 Let $A = (a_{ij}) \in M_{m \times n}$ be $m \times n$ matrix and $s, t \in \mathbb{N}$ be natural numbers such that $1 \leq s \leq m$, $1 \leq t \leq n$. We define the map $\underline{cst}$ from $M_{m \times n}$ to $M_{s \times t}$ by corresponding to the matrix A , the matrix $A \underline{cst} = (a_{ij})$ where $1 \leq i \leq s$, $1 \leq j \leq t$. We call this map *cut-projection* of type $\underline{st}$.

We use cut-projections on all types of matrices to define sums and products.

Definition 3.2 Let $A = (a_{ij}) \in M_{m \times n}$ be an $m \times n$ matrix and $s, t \in \mathbb{N}$, such that $1 \leq s \leq m$, $1 \leq t \leq n$. We define the mod-like map $\underline{st}$ from $M_{m \times n}$ to $M_{s \times t}$ by corresponding to A the matrix $A \underline{st} = (a_{ij})$ which has as entries the sets

$$\underline{a}_{ij} = \{a_{i+\kappa s, j+\lambda t} \mid 1 \leq i \leq s, 1 \leq j \leq t, \text{ and } \kappa, \lambda \in \mathbb{N}, i+\kappa s \leq m, j+\lambda t \leq n\}.$$

We call *helix-projection* of type $\underline{st}$, the multivalued map

$$\underline{st} : M_{m \times n} \rightarrow M_{s \times t} : A \rightarrow A \underline{st} = (\underline{a}_{ij}).$$

$A \underline{st}$ is a set of $s \times t$ -matrices $X = (x_{ij})$ such that $\forall i, j$ we have $x_{ij} \in \underline{a}_{ij}$.

Let $A = (a_{ij}) \in M_{m \times n}$ be a matrix and $s, t \in \mathbb{N}$ such that $1 \leq s \leq m$, $1 \leq t \leq n$. It is clear that, we can apply the helix-projection first on the rows and then on the columns, the result is the same if we apply the helix-projection on both, rows and columns. Therefore, we have

$$(A_{\underline{s}n})_{\underline{s}t} = (A_{\underline{m}t})_{\underline{s}t} = A_{\underline{s}t}.$$

Let $A=(a_{ij}) \in M_{m \times n}$ be matrix and $s, t \in \mathbb{N}$ such that $1 \leq s \leq m$, $1 \leq t \leq n$. Then if $A_{\underline{s}t}$ is not a set but one single matrix then we call A , *cut-helix matrix* of type $s \times t$. In other words the matrix A is a helix matrix of type $s \times t$, if $A_{\underline{s}t} = A_{\underline{s}t}$.

Definition 3.3

- a. Let $A=(a_{ij}) \in M_{m \times n}$, $B=(b_{ij}) \in M_{u \times v}$ be matrices and $s=\min(m, u)$, $t=\min(n, v)$. We define a hope, called *helix-addition* or *helix-sum*, as follows:

$$\oplus: M_{m \times n} \times M_{u \times v} \rightarrow P(M_{s \times t}): (A, B) \rightarrow A \oplus B = A_{\underline{s}t} + B_{\underline{s}t} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset M_{s \times t},$$

where

$$(\underline{a}_{ij}) + (\underline{b}_{ij}) = \{(c_{ij}) = (a_{ij} + b_{ij}) \mid a_{ij} \in \underline{a}_{ij} \text{ and } b_{ij} \in \underline{b}_{ij}\}.$$

- b. Let $A=(a_{ij}) \in M_{m \times n}$ and $B=(b_{ij}) \in M_{u \times v}$ be matrices and $s=\min(n, u)$. We define a hope, called *helix-multiplication* or *helix-product*, as follows:

$$\otimes: M_{m \times n} \times M_{u \times v} \rightarrow P(M_{m \times v}): (A, B) \rightarrow A \otimes B = A_{\underline{m}s} \cdot B_{\underline{s}v} = (\underline{a}_{ij}) \cdot (\underline{b}_{ij}) \subset M_{m \times v},$$

$$\text{where } (\underline{a}_{ij}) \cdot (\underline{b}_{ij}) = \{(c_{ij}) = (\sum a_{it} b_{tj}) \mid a_{ij} \in \underline{a}_{ij} \text{ and } b_{ij} \in \underline{b}_{ij}\}.$$

The helix-sum is an external hope and the commutativity is valid.

Remark. In $M_{m \times n}$ the addition is ordinary operation, thus we are interested only in the 'product'. From the fact that the helix-product on non-square matrices is defined, the definition of the Lie-bracket is immediate, so the *helix-Lie Algebra* is defined, as well. This is an H_v -Lie Algebra where the relation ε^* gives, by a quotient, a Lie algebra.

In the following we restrict ourselves on the matrices $M_{m \times n}$ where $m < n$. We have analogous results if $m > n$ and for $m = n$ we have the classical theory.

Notation. For given $\kappa \in \mathbb{N} - \{0\}$, we denote by $\underline{\kappa}$ the remainder resulting from its division by m if the remainder is non zero, and $\underline{\kappa} = m$ if the remainder is zero. Thus a matrix $A = (a_{\kappa\lambda}) \in M_{m \times n}$, $m < n$ is a *cut-helix matrix* if we have $a_{\kappa\lambda} = a_{\kappa\lambda}$, $\forall \kappa, \lambda \in \mathbb{N} - \{0\}$.

Moreover, let us denote by $I_c = (c_{\kappa\lambda})$ the *cut-helix unit matrix* which the cut matrix is the unit matrix I_m . Therefore, since $I_m = (\delta_{\kappa\lambda})$, where $\delta_{\kappa\lambda}$ is the Kronecker's delta, we obtain that, $\forall \kappa, \lambda$, $c_{\kappa\lambda} = \delta_{\kappa\lambda}$.

Proposition. For $m < n$ in $(M_{m \times n}, \otimes)$ the cut-helix unit matrix $I_c = (c_{\kappa\lambda})$, where $c_{\kappa\lambda} = \delta_{\kappa\lambda}$, is a left scalar unit and a right unit. It is the only one left scalar unit.

Definition 3.4 Let $A=(a_{ij}) \in M_{m \times n}$ be matrix and $s, t \in \mathbb{N}$ such that $1 \leq s \leq m$, $1 \leq t \leq n$. Then if A_{st} is a set of upper triangular matrices with the same diagonal, then we call A an **S-helix matrix of type $s \times t$** . Therefore, in an S-helix matrix A of type $s \times t$, the A_{st} has on the diagonal entries which are not sets but elements.

Proposition 3.5 The set of S-helix matrices $A=(a_{ij}) \in M_{m \times n}$ with $m < n$, is closed under the helix product. Moreover, it has a unit the cut-helix unit matrix I_c , which is left scalar.

Proof. The helix product of two S-helix matrices, $X=(x_{ij}), Y=(a_{ij}) \in M_{m \times n}$, $X \otimes Y$, contain matrices $Z=(z_{ij})$, which are upper diagonals. Moreover, for every z_{ii} , the entry ii is singleton since it is product of only $z_{(i+km), (i+km)} = z_{ii}$, entries. The unit is the matrix $I_c = I_{m \times n}$, where we have $I_{m \times n} = I_{mm} = I_m$.

Definition 3.6 We call a matrix $A=(a_{ij}) \in M_{m \times n}$ **S_0 -helix matrix** if it is S-helix matrix where the condition $a_{11}a_{22} \dots a_{mm} \neq 0$, is valid. Thus, an S_0 -helix matrix has no zero elements on the diagonal and S_0 is subset of S of all S-helix matrices. Notice that this set is closed under the helix product not in addition. Therefore, it is interesting only when the product is used not the addition.

Proposition 3.7 The set of S_0 -helix matrices $A=(a_{ij}) \in M_{m \times n}$ with $m < n$, is closed under the helix product, it has a unit the cut-helix unit matrix I_c , which is left scalar and S_0 -helix matrices X have inverses X^{-1} , i.e. $I_c \in X \otimes X^{-1} \cap X^{-1} \otimes X$.

On small finite H_V -fields important cases appear taking the generating sets:

Example 3.8 On the type 3×5 of matrices using the $(\mathbb{Z}_4, +, \cdot)$ where we take the small H_V -field $(\mathbb{Z}_4, +, \otimes)$, where only $2 \otimes 3 = \{0, 2\}$ and fundamental classes $\{0, 2\}$, $\{1, 3\}$. We consider the set of all S_0 -helix matrices and we take the S_0 -helix matrix:

$$X = \begin{pmatrix} 1 & 2 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

We obtain that the generating set is the following

$$\begin{pmatrix} 1 & \{0, 2\} & \{0, 2\} & 1 & \{0, 2\} \\ 0 & \{1, 3\} & \{0, 1\} & 0 & \{1, 3\} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

where in the 22 and 25 entries appears simultaneously 1 or 3.

An example of hyper-matrix representation, 7-dimensional, with helix-hope:

Example 3.9 Consider the matrices of type 3×5 on the field of complex numbers, of the type

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{11} & x_{15} \\ 0 & x_{22} & x_{23} & 0 & x_{22} \\ 0 & 0 & x_{33} & 0 & 0 \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{11} & y_{15} \\ 0 & y_{22} & y_{23} & 0 & y_{22} \\ 0 & 0 & y_{33} & 0 & 0 \end{pmatrix}$$

then we have

$$X \otimes Y = \begin{pmatrix} x_{11} & \{x_{12}, x_{15}\} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} \cdot \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{11} & y_{15} \\ 0 & y_{22} & y_{23} & 0 & y_{22} \\ 0 & 0 & y_{33} & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} x_{11}y_{11} & x_{11}y_{12} + \{x_{12}, x_{15}\}y_{22} & x_{11}y_{13} + \{x_{12}, x_{15}\}y_{23} + x_{13}y_{33} & x_{11}y_{11} & x_{11}y_{15} + \{x_{12}, x_{15}\}y_{22} \\ 0 & x_{22}y_{22} & x_{22}y_{23} + x_{23}y_{33} & 0 & x_{22}y_{22} \\ 0 & 0 & x_{33}y_{33} & 0 & 0 \end{pmatrix}$$

Therefore, the helix product is a set with cardinality up to 8.

The unit of this type is $I_c = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

4. THE HYPER LIE-SANTILLI ADMISSIBILITY

The general definition of the Lie-Santilli admissibility on hyperstructures is [2], [4], [5], [6], [7], [9], [10], [20], [22], [23], [24], [33]:

Definition 4.1 Let L be an H_V -vector space over the H_V -field $(F, +, \cdot)$, $\varphi: F \rightarrow F/\gamma^*$, the canonical map and $\omega_F = \{x \in F: \varphi(x) = 0\}$, where 0 is the zero of the fundamental field F/γ^* . Moreover, let ω_L be the core of the canonical map $\varphi': L \rightarrow L/\varepsilon^*$ and denote by the same symbol 0 the zero of L/ε^* . Take any two subsets $R, S \subseteq L$ then the *Lie-Santilli admissible hyperalgebra* is obtained by taking the Lie bracket, which is the hope:

$$[,]_{RS}: L \times L \rightarrow P(L): [x, y]_{RS} = xRy - ySx = \{xry - ysx \mid r \in R, s \in S\}$$

Special case but not degenerate, are the 'small' and 'strict' ones:

- (a) When only $\mathcal{S}$ is considered, then $[x,y]_{\mathcal{S}} = xy - ySx = \{xy - ysx \mid s \in \mathcal{S}\}$
 (b) When only $\mathcal{R}$ is considered, then $[x,y]_{\mathcal{R}} = xRy - yx = \{xry - yx \mid r \in \mathcal{R}\}$
 (c) We have the 'like' P-hope for any subset $\mathcal{S} \subseteq \mathcal{L}$: $[x,y]_{\mathcal{S}} = \{xsy - ysx \mid s \in \mathcal{S}\}$

Example 4.2 In the set of 2×2 traceless complex matrices, take the self inverse matrices

$$\mathcal{R} = \mathcal{S} = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

Then, take two traceless matrices

$$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & -y_1 \end{pmatrix}$$

the Lie bracket is $[X,Y]_{\mathcal{R}\mathcal{R}} = XRY - YRX =$

$$\left\{ \begin{pmatrix} x_2y_1 + x_1y_3 - x_1y_2 - x_3y_1 & 0 \\ 0 & x_2y_1 + x_1y_3 - x_1y_2 - x_3y_1 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} -x_1y_1 + x_3y_3 - x_1y_3 - x_3y_1 & -x_1y_2 - x_3y_1 - x_2y_3 + x_1y_1 \\ x_2y_1 + x_1y_3 - x_1y_1 + x_3y_2 & x_2y_2 - x_1y_1 - x_2y_1 - x_1y_2 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} x_1y_1 - x_2y_3 - x_1y_2 - x_3y_1 & x_1y_2 + x_2y_1 - x_2y_2 + x_1y_1 \\ x_2y_1 + x_1y_3 + x_1y_1 - x_3y_3 & x_3y_2 - x_1y_1 + x_2y_1 + x_1y_3 \end{pmatrix}, \begin{pmatrix} -x_2y_3 + x_3y_2 & 0 \\ 0 & x_3y_2 - x_2y_3 \end{pmatrix} \right\}$$

Therefore, the result is a set of 4 matrices, where the last one is traceless.

The results simplified in cases as the upper triangular traceless matrices.

Example 4.3 In the set of 3×3 traceless complex matrices of the form

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ 0 & -x_1 - 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } R = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

Then, if we denote

$$q_1 = -x_1y_2 + x_2y_1 + 2x_2 - 2y_2, \quad q_2 = -x_1y_2 + x_2y_1 + x_1y_1 + y_1 + 2x_2 - 2y_2, \\ q_3 = -x_1y_2 + x_2y_1 - x_1y_1 + x_1 + 2x_2 - 2y_2, \quad q_4 = -x_1y_2 + x_2y_1 - x_1 + y_1 + 2x_2 - 2y_2,$$

we obtain that the Lie bracket is the following

$$[X, Y]_{RR} = XRY - YRX = \left\{ \begin{pmatrix} 0 & q_i & x_1y_3 + x_3y_1 - x_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : i=1,2,3,4 \right\}$$

The case of S-helix-hope on real or complex matrices of type 3×6

Denote E_{ij} the $m \times n$ matrix where the ij entry is 1 and the rest entries are 0. Consider the set of 3×6 matrices of type

$$X = x_{11}E_{11} + x_{12}E_{12} + x_{13}E_{13} + x_{14}E_{14} + x_{15}E_{15} + x_{16}E_{16} \\ + x_{22}E_{22} + x_{23}E_{23} + x_{25}E_{25} + x_{26}E_{26} + x_{33}E_{33} + x_{36}E_{36}$$

which is a 9-dimentional space.

Notice that the 3×6 cut-helix matrices are

$$X_{ch} = x_{11}E_{11} + \{x_{12}, x_{15}\}E_{12} + \{x_{13}, x_{16}\}E_{13} + x_{22}E_{22} + \{x_{23}, x_{26}\}E_{23} + x_{33}E_{33}$$

The helix-product of two matrices X and Y is the set of matrices given by:

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{11} & x_{15} & x_{16} \\ 0 & x_{22} & x_{23} & 0 & x_{22} & x_{26} \\ 0 & 0 & x_{33} & 0 & 0 & x_{33} \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{11} & y_{15} & y_{16} \\ 0 & y_{22} & y_{23} & 0 & y_{22} & y_{26} \\ 0 & 0 & y_{33} & 0 & 0 & y_{33} \end{pmatrix}$$

$$X \otimes Y = \begin{pmatrix} x_{11} & \{x_{12}, x_{15}\} & \{x_{13}, x_{16}\} \\ 0 & x_{22} & \{x_{23}, x_{26}\} \\ 0 & 0 & x_{33} \end{pmatrix} \cdot \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{11} & y_{15} & y_{16} \\ 0 & y_{22} & y_{23} & 0 & y_{22} & y_{26} \\ 0 & 0 & y_{33} & 0 & 0 & y_{33} \end{pmatrix} = M =$$

$$= x_{11}y_{11}E_{11} + (x_{11}y_{12} + \{x_{12}, x_{15}\}y_{22})E_{12} + (x_{11}y_{13} + \{x_{12}, x_{15}\}y_{23} + \{x_{13}, x_{16}\}y_{33})E_{13} \\ + x_{11}y_{14}E_{14} + (x_{11}y_{15} + \{x_{12}, x_{15}\}y_{22})E_{15} + (x_{11}y_{16} + \{x_{12}, x_{15}\}y_{26} + \{x_{13}, x_{16}\}y_{33})E_{16} \\ + x_{22}y_{22}E_{22} + (x_{22}y_{23} + \{x_{23}, x_{26}\}y_{33})E_{23} + x_{22}y_{25}E_{25} + (x_{22}y_{26} + \{x_{23}, x_{26}\}y_{33})E_{26} \\ + x_{33}y_{33}E_{33} + x_{33}y_{36}E_{36}$$

Therefore, the helix product is a set with cardinality up to 2^8 . The 3-dimensions have single-valued results and 6-dimensions multivalued.

There are subspaces of the above, which reduce the cardinality of results. One has to check if the set of S-helix matrices is closed to the helix-product.

The dimensions on the multivalued cases can be reduced two-dimensions if we take $x_{12}=x_{15}$ or $x_{23}=x_{26}$ then we have 8-dimensional subspace. In this case the single valued and multivalued subspaces are of 4-dimensions.

8-dimensional case

Consider two S-helix matrices X and Y of the type 3×6 on the field of real or complex numbers where the 12 and 15 entries are the same. Suppose

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{11} & x_{12} & x_{16} \\ 0 & x_{22} & x_{23} & 0 & x_{22} & x_{26} \\ 0 & 0 & x_{33} & 0 & 0 & x_{33} \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{11} & y_{12} & y_{16} \\ 0 & y_{22} & y_{23} & 0 & y_{22} & y_{26} \\ 0 & 0 & y_{33} & 0 & 0 & y_{33} \end{pmatrix},$$

then we obtain

$$\begin{aligned} X \otimes Y &= \begin{pmatrix} x_{11} & x_{12} & \{x_{13}, x_{16}\} \\ 0 & x_{22} & \{x_{23}, x_{26}\} \\ 0 & 0 & x_{33} \end{pmatrix} \cdot \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{11} & y_{12} & y_{16} \\ 0 & y_{22} & y_{23} & 0 & y_{22} & y_{26} \\ 0 & 0 & y_{33} & 0 & 0 & y_{33} \end{pmatrix} = \\ &= x_{11}y_{11}E_{11} + (x_{11}y_{12}+x_{12}y_{22})E_{12} + (x_{11}y_{13}+x_{12}y_{23}+\{x_{13}, x_{16}\}y_{33})E_{13} \\ &+ x_{11}y_{11}E_{14} + (x_{11}y_{12}+x_{12}y_{22})E_{15} + (x_{11}y_{16}+x_{12}y_{26}+\{x_{13}, x_{16}\}y_{33})E_{16} \\ &+ x_{22}y_{22}E_{22} + (x_{22}y_{23}+\{x_{23}, x_{26}\}y_{33})E_{23} + x_{22}y_{22}E_{25} + (x_{22}y_{26}+\{x_{23}, x_{26}\}y_{33})E_{26} \\ &+ x_{33}y_{33}E_{33} + x_{33}y_{33}E_{36} \end{aligned}$$

Therefore, the helix product is a set with cardinality up to 2^4 .

The dimensions on the multivalued cases can be reduced four-dimensions if we take $x_{12}=x_{15}$ and $x_{23}=x_{26}$ then we have 7-dimensional subspace. The single valued subspace is of 5-dimension and the multivalued of 2-dimensions. Take

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{11} & x_{12} & x_{16} \\ 0 & x_{22} & x_{23} & 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} & 0 & 0 & x_{33} \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{11} & y_{12} & y_{16} \\ 0 & y_{22} & y_{23} & 0 & y_{22} & y_{23} \\ 0 & 0 & y_{33} & 0 & 0 & y_{33} \end{pmatrix},$$

then we obtain

$$\begin{aligned}
 X \otimes Y &= \begin{bmatrix} x_{11} & x_{12} & \{x_{13}, x_{16}\} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix} \cdot \begin{bmatrix} y_{11} & y_{12} & y_{13} & y_{11} & y_{12} & y_{16} \\ 0 & y_{22} & y_{23} & 0 & y_{22} & y_{23} \\ 0 & 0 & y_{33} & 0 & 0 & y_{33} \end{bmatrix} = \\
 &= x_{11}y_{11}E_{11} + (x_{11}y_{12}+x_{12}y_{22})E_{12} + (x_{11}y_{13}+x_{12}y_{23}+\{x_{13}, x_{16}\}y_{33})E_{13} + x_{11}y_{11}E_{14} \\
 &+ (x_{11}y_{12}+x_{12}y_{22})E_{15} + (x_{11}y_{16}+x_{12}y_{23}+\{x_{13}, x_{16}\}y_{33})E_{16} + x_{22}y_{22}E_{22} \\
 &+(x_{22}y_{23}+x_{23}y_{33})E_{23} + x_{22}y_{22}E_{25} + (x_{22}y_{23}+x_{23}y_{33})E_{26} + x_{33}y_{33}E_{33} + x_{33}y_{33}E_{36}
 \end{aligned}$$

Therefore, the helix product is a set with cardinality up to 2^4 .

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ON THE SANTILLI'S ISO-HYPER-MATHEMATICS.
THE SANTILLI'S HYPER-NUMBERS.

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Abstract

The Lie-Santilli theory on '*isotopies*' and '*genotopies*' was born in 1960's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting' of the n -dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n -dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1996, and they are called *e-hyperfields*. The H_v -fields can give *e-hyperfields* which can be used in the isotopy theory in applications as in physics or biology. In fact the main question in this theory proposed by Professor Santilli is: What are the hypernumbers? These are elements of special *e-hyperfields* mainly from H_v -fields. We present special constructions on those hyperfields in order to answer new questions on isotopies theory especially on the, so called, *fundamentally small very thin H_v -fields*.

Key words: Lie-Santilli iso-theory, hyperstructures, hope, H_v -structures

AMS Subject Classification: 20N20, 16Y99

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1. INTRODUCTION AND BASIC DEFINITIONS

In [24] there is a first description on how Santilli's theories effect in hyperstructures and how new theories in Mathematics appeared by Santilli's pioneer research. We continue with new topics in this direction.

Last decades hyperstructures have applications in mathematics and in other sciences, which range from biomathematics -conchology, inheritance- and hadronic physics or on leptons, in the Santilli's iso-theory, to mention but a few. The hyperstructure theory is closely related to fuzzy theory; consequently, can be widely applicable in linguistic, in sociology, in industry and production, too. For these applications the largest class of the hyperstructures, the class *H_v-structures*, is used, they satisfy the *weak axioms* where the non-empty intersection replaces the equality. The main tools of this theory are the *fundamental relations* which connect, by quotients, the H_v-structures with the corresponding classical ones. These relations are used to define hyperstructures as H_v-fields, H_v-vector spaces and so on, as well. *Hypernumbers or H_v-numbers* are called the elements of an H_v-field.

The hyperstructures were introduced by F. Marty in 1934, when he defined the hypergroup as a set equipped with an associative and reproductive hyperoperation. The motivating example was the quotient of a group by any, not necessarily normal, subgroup. M. Koskas in 1970 introduced the relation β^* , which it turns to be the main tool in the study of hyperstructures. T. Vougiouklis in 1990 [14] was introduced the H_v-structures, by defining the weak axioms. The motivating example of those hyperstructures is the quotient of any group by any partition.

In 1996 R. M. Santilli & T. Vougiouklis [8], point out that in physics the most interesting hyperstructures are the one called e-hyperstructures. These hyperstructures contain a unique left and right scalar unit, which is used in Lie-Santilli theory. In what follows we present the related hyperstructure theory with some new results. However, one can see books and related papers for more definitions, results and applications: [1], [2], [3], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [22], [23], [27], [32], [33].

In a set H is called *hyperoperation* (abbreviated by *hope*) or *multivalued operation*, any map from $H \times H$ to the power set of H. Therefore, in a hope

$$\cdot : H \times H \rightarrow \wp(H): (x,y) \rightarrow x \cdot y \subset H$$

the result is subset of H, instead of an element as we have in operations.

In a set H equipped with a hope $\cdot : H \times H \rightarrow P(H) - \{\emptyset\}$, we abbreviate by

WASS the *weak associativity*: $(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H$ and by

COW the *weak commutativity*: $xy \cap yx \neq \emptyset, \forall x, y \in H$.

The hyperstructure $(H, \cdot)$ is called *H_v-semigroup* if it is *WASS* and it is called *H_v-group* if it is reproductive *H_v-semigroup*, i.e. $xH = Hx = H, \forall x \in H$. The hyperstructure $(R, +, \cdot)$ is called *H_v-ring* if $(+)$ and $(\cdot)$ are *WASS*, the reproduction axiom is valid for $(+)$, and $(\cdot)$ is *weak distributive* to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \emptyset, (x+y)z \cap (xz+yz) \neq \emptyset, \forall x, y, z \in R.$$

An extreme class of hyperstructures is the following [13]: An *H_v-structure* is *very thin* iff all hopes are operations except one, which has all hyperproducts singletons except one, which is a subset of cardinality more than one.

An *H_v-group* is called *cyclic* [16], if there is an element, called *generator*, which the powers have union the underline set. The minimal power with the above property is called *period* of the generator. Moreover, if there exist an element and a special power, the minimum one, is the underline set, then the *H_v-group* is called *single-power cyclic*.

The main tool to study all hyperstructures are the fundamental relations β^* , γ^* and ε^* , which are defined, in *H_v-groups*, *H_v-rings* and *H_v-vector spaces*, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively [16].

Theorem 1.1 Let $(H, \cdot)$ be an *H_v-group* and U be all finite products of elements of H . We define the relation β by setting $x\beta y$ iff $\{x, y\} \subset u, u \in U$. Then β^* is the transitive closure of β .

We present a short proof for the analogous to above theorem in the case of *H_v-rings*:

Theorem 1.2 Let $(R, +, \cdot)$ be *H_v-ring*, U the set of finite polynomials of elements of R . Define the relation γ in R as follows: $x\gamma y$ iff $\{x, y\} \subset u$ where $u \in U$. Then γ^* is the transitive closure of γ .

Proof. Let γ be transitive closure of γ , and $\gamma(a)$ the class of a . First, we prove that M/γ is ring.

In R/γ the sum $(\oplus)$ and the product $(\otimes)$ are defined in the usual manner:

$$\gamma(a) \oplus \gamma(b) = \{\gamma(c) : c \in \gamma(a) + \gamma(b)\}, \gamma^*(a) \otimes \gamma(b) = \{\gamma(d) : d \in \gamma^*(a) \cdot \gamma(b)\}, \forall a, b \in R.$$

Take $a' \in \gamma(a)$, $b' \in \gamma(b)$. Then we have

$$\begin{aligned} a' \gamma a & \text{ iff } \exists x_1, \dots, x_{m+1} \text{ with } x_1 = a', x_{m+1} = a \text{ and } u_1, \dots, u_m \in U: \{x_i, x_{i+1}\} \subset u_i, \\ & \quad i=1, \dots, m, \text{ and} \\ b' \gamma b & \text{ iff } \exists y_1, \dots, y_{n+1} \text{ with } y_1 = b', y_{n+1} = b \text{ and } v_1, \dots, v_n \in U: \{y_j, y_{j+1}\} \subset v_j, \\ & \quad j=1, \dots, n. \end{aligned}$$

From the above we obtain

$$\{x_i, x_{i+1}\} + y_1 \subset u_1 + v_1, i=1, \dots, m-1, x_{m+1} + \{y_j, y_{j+1}\} \subset u_m + v_j, j=1, \dots, n.$$

The $u_i + v_1 = t_i$, $i=1, \dots, m-1$ and $u_m + v_j = t_{m+j-1}$, $j=1, \dots, n$, are polynomials, so $t_k \in U$, $\forall k \in \{1, \dots, m+n-1\}$.

Now, pick up $z_1, \dots, z_{m+n}$: $z_i \in x_i + y_1$, $i=1, \dots, n$ and $z_{m+j} \in x_{m+1} + y_{j+1}$, $j=1, \dots, n$, thus using the above relations we obtain $\{z_k, z_{k+1}\} \subset t_k$, $k=1, \dots, m+n-1$.

So, every element $z_1 \in x_1 + y_1 = a' + b'$ is γ equivalent to $z_{m+n} \in x_{m+1} + y_{n+1} = a + b$. Therefore, $\gamma(a) \oplus \gamma(b)$ is singleton and we write $\gamma(a) \oplus \gamma(b) = \gamma(c)$, $\forall c \in \gamma(a) + \gamma(b)$.

Similarly, we prove that $\gamma(a) \otimes \gamma(b) = \gamma(d)$, $\forall d \in \gamma(a) \cdot \gamma(b)$.

The WASS and the weak distributivity on R guarantee that the associativity and the distributivity are valid for the quotient R/γ^* . Thus, R/γ^* is a ring.

Now let σ be equivalence relation in R such that R/σ is a ring, denote $\sigma(a)$ the class of a . Then $\sigma(a) \oplus \sigma(b)$ and $\sigma(a) \otimes \sigma(b)$ are singletons for all $a, b \in R$, i.e.

$$\sigma(a) \oplus \sigma(b) = \sigma(c), \forall c \in \sigma(a) + \sigma(b), \sigma(a) \otimes \sigma(b) = \sigma(d), \forall d \in \sigma(a) \cdot \sigma(b).$$

Therefore, we can write, $\forall a, b \in R$ and $A \subset \sigma(a)$, $B \subset \sigma(b)$,

$$\sigma(a) \oplus \sigma(b) = \sigma(a+b) = \sigma(A+B), \sigma(a) \otimes \sigma(b) = \sigma(ab) = \sigma(A \cdot B).$$

By induction, we extend these relations on finite sums and products. Thus, for every $u \in U$, we have the relation $\sigma(x) = \sigma(u)$, $\forall x \in u$. Consequently, $x \in \gamma(a)$ implies $x \in \sigma(a)$, $\forall x \in R$.

But σ is transitively closed, so we obtain: $x \in \gamma(x)$ implies, $x \in \sigma(a)$.

That means that γ is the smallest equivalence relation in R such that R/γ is a ring, i.e. $\gamma = \gamma^*$. ■

An element is called *single* if its fundamental class is singleton.

The fundamental relations are used for general definitions. Thus, in order to define the H_v -field the γ^* is used [14]: An H_v -ring $(R, +, \cdot)$ is called **H_v -field** if R/γ^* is a field. In the sequence the H_v -vector space is defined.

Let $(H, \cdot)$, $(H, *)$ be H_v -semigroups defined on the same set H . The hope $(\cdot)$ is called *smaller* than $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an

$$f \in \text{Aut}(H, *) \text{ such that } xy \subset f(x*y), \forall x, y \in H.$$

Then we write $\cdot \leq^*$ and we say that $(H, *)$ *contains* $(H, \cdot)$. If $(H, \cdot)$ is a structure then it is called *basic structure* and $(H, *)$ is called *H_b -structure*.

The little Theorem 1.3 Greater hopes than the ones which are *WASS* or *COW*, are also *WASS* or *COW*, respectively.

The definition of H_v -field introduced a new class of hyperstructures [22]:

Definition 1.4 The H_v -semigroup $(H, \cdot)$ is called *h/v -group* if the quotient H/β^* is a group.

The h/v -groups are a generalization of the H_v -groups since in h/v -groups the reproductivity is not necessarily valid. Sometimes a kind of *reproductivity of classes* is valid. This leads the quotient to be reproductivity. In similar way, the *h/v -rings*, *h/v -fields*, *h/v -modulus*, *h/v -vector spaces* etc, are defined.

Definition 1.5 [16]. Let $(F, +, \cdot)$ be H_v -field, $(V, +)$ a *COW* H_v -group and there exists an external hope

$$\cdot : F \times V \rightarrow P(V): (a, x) \rightarrow ax$$

such that, $\forall a, b \in F$ and $\forall x, y \in V$, we have

$$a(x+y) \cap (ax+ay) \neq \emptyset, (a+b)x \cap (ax+bx) \neq \emptyset, (ab)x \cap a(bx) \neq \emptyset,$$

then V is called *H_v -vector space* over F .

In the case of an H_v -ring instead of H_v -field then the *H_v -modulo* is defined.

In the above cases the fundamental relation ε^* is the smallest equivalence such that the quotient V/ε^* is a vector space over the fundamental field F/γ^* .

The general definition of an H_v -Lie algebra is the following [26], [27], [33]:

Definition 1.6 Let $(L, +)$ be H_v -vector space on $(F, +, \cdot)$, $\varphi: F \rightarrow F/\gamma^*$, canonical map and $\omega_F = \{x \in F: \varphi(x) = 0\}$, where 0 is the zero of the fundamental field F/γ^* . Similarly, let ω_L be the core of the canonical map $\varphi': L \rightarrow L/\varepsilon^*$ and denote by the same symbol 0 the zero of L/ε^* . Consider the bracket hope (commutator):

$$[,] : L \times L \rightarrow P(L): (x, y) \rightarrow [x, y]$$

then L is an *H_v -Lie algebra* over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e. $\forall x, x_1, x_2, y, y_1, y_2 \in L$ and $\forall \lambda_1, \lambda_2 \in F$

$$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \emptyset, [x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \emptyset$$

$$(L2) \quad [x, x] \cap \omega_L \neq \emptyset, \quad \forall x \in L$$

$$(L3) \quad ([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \emptyset, \quad \forall x, y \in L$$

This is general definition thus special cases can be used, in order to face problems in applied sciences. Moreover, we see how the weak properties can be defined as the above weak linearity (L1), anti-commutativity (L2) and the Jacobi identity (L3). Similarly, the *h/v-rings*, *h/v-fields*, *h/v-modulus*, *h/v-vector spaces* etc, are defined.

2. ENLARGE AND REDUCE HOPES. ∂ -HOPES, P-HOPES

The 'enlarged' hyperstructures [19] were examined in the sense that an extra element, outside the underlying set, appears in one result. In the 'reduced' case in fact we exclude elements. In enlargement or reduction most useful in representation theory, are those H_v -structures with the same fundamental structure: Suppose we have a structure and one element, outside of the structure, then we can attach this element to obtain a hyperstructure which is an h/v -structure. The opposite problem is: Can we remove at least one element of an H_v -structure or a classical structure and obtain analogous h/v -structure?

The Attach Construction 2.1 Let $(H, \cdot)$ be an H_v -semigroup and $v \notin H$. We extend the $(\cdot)$ into the $\underline{H} = H \cup \{v\}$ as follows: $x \cdot v = v \cdot x = v, \forall x \in H$, and $v \cdot v = H$.

The $(\underline{H}, \cdot)$ is an h/v -group where $(\underline{H}, \cdot) / \beta^* \cong Z_2$ and v is a single element.

We call the hyperstructure $(\underline{H}, \cdot)$ the *attach h/v -group* of $(H, \cdot)$.

In the representation theory of H_v -groups by H_v -matrices one needs H_v -rings or H_v -fields which have non-degenerate fundamental structures in addition with only few of hypersums and hyperproducts have cardinals greater than one.

Theorem 2.2 Let $(G, \cdot)$ be semigroup and $v \notin G$ be an element appearing in a product ab , where $a, b \in G$, thus the result becomes a hyperproduct $a \otimes b = \{ab, v\}$. Then the minimal hope $(\otimes)$ extended in $G' = G \cup \{v\}$ such that $(\otimes)$ contains $(\cdot)$ in the restriction on G , and such that $(G', \otimes)$ is a minimal H_v -semigroup which has fundamental structure isomorphic to $(G, \cdot)$, is defined as follows:

$$a \otimes b = \{ab, v\}, \quad x \otimes y = xy, \quad \forall (x, y) \in G^2 - \{(a, b)\}$$

$$v \otimes v = abab, \quad x \otimes v = xab \quad \text{and} \quad v \otimes x = abx, \quad \forall x \in G.$$

Therefore, $(G', \otimes)$ is a very thin H_v -semigroup.

If $(G, \cdot)$ is commutative then the $(G', \otimes)$ becomes strongly commutative.

Definitions 2.3 Let $(H, \cdot)$ be a hypergroupoid.

We say that *remove* $h \in H$, if simply consider the restriction of $(\cdot)$ on $H - \{h\}$.

We say that $\underline{h} \in H$ *absorbs* $h \in H$ if we replace h , whenever it appears, by $\underline{h}$.

We say that $\underline{h} \in H$ *merges* with $h \in H$, if we take as product of $x \in H$ by $\underline{h}$, the union of the results of x with both h and $\underline{h}$, and consider h and $\underline{h}$ as one class.

The *uniting elements* method was introduced by Corsini & Vougiouklis [2] in 1989. With this method one puts in the same class, two or more elements and, through hyperstructures, we obtain structures with additional properties.

The *uniting elements* method is the following: Let G be algebraic structure and let d be a property, which is not valid, and it is described by a set of equations; then, consider the partition in G for which it is put together, in the same partition class, every pair of elements that causes the non-validity of the property d . The quotient by this partition G/d is an H_V -structure. Then, quotient out the H_V -structure G/d by the fundamental relation β^* , a stricter structure, $(G/d)/\beta^*$, for which the property d is valid, is obtained.

An application is when more than one properties are desired. Some of the properties lead straight to the classes than others, thus it is better to apply first them followed by the more complicated ones. We can do this because the following is valid:

Theorem 2.4 Let $(G, \cdot)$ be groupoid, and $F = \{f_1, \dots, f_m, f_{m+1}, \dots, f_{m+n}\}$ be system of equations on G consisting of two systems $F_m = \{f_1, \dots, f_m\}$, $F_n = \{f_{m+1}, \dots, f_{m+n}\}$. Let σ , σ_m be the equivalences defined by the uniting elements procedure using the systems F and F_m resp., and let σ_n be the equivalence relation defined using the induced equations of F_n on the groupoid $G_m = (G/\sigma_m)/\beta^*$. Then

$$(G/\sigma)/\beta^* \cong (G_m/\sigma_n)/\beta^*.$$

From the above it is clear that the fundamental structure is very important, mainly if it is known from the beginning. This is the problem to construct hyperstructures with desired fundamental structures. Examples how to obtain properties as associativity, commutativity, reproductivity can be found in [18].

In [23] a hope, in a groupoid with a map on it, called *theta* ∂ , is introduced.

Definitions 2.5 Let $(G, \cdot)$ be groupoid (resp., hypergroupoid) and $f: G \rightarrow G$ be map. We define a hope (∂) , called *theta* and we write *∂ -hope*, on G as follows

$$x\partial y = \{f(x) \cdot y, x \cdot f(y)\}, \forall x, y \in G. \text{ (resp. } x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \forall x, y \in G)$$

If $(\cdot)$ is commutative then (∂) is commutative. If $(\cdot)$ is *COW*, then (∂) is *COW*.
Let $(G, \cdot)$ a groupoid (resp. hypergroupoid) and $f: G \rightarrow P(G) - \{\emptyset\}$ multivalued map. We define the (∂) , on G as follows

$$x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \forall x, y \in G.$$

Let $(G, \cdot)$ be a groupoid, $f_i: G \rightarrow G$, $i \in I$, be a set of maps on G . The

$$f \cup: G \rightarrow P(G): f \cup(x) = \{f_i(x) \mid i \in I\},$$

is the union of $f_i(x)$. We have the *union theta-hope* (∂) , on G if we take $f \cup(x)$.
If we take $\underline{f} = f \cup(\text{id})$, then we have the *b-theta-hope*: $\underline{\partial}$.

This definition can be generalized for more operations or hopes.

Motivation for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. Therefore, in these terms, for two functions $s(x)$, $t(x)$, we have $s\partial t = \{s't, st'\}$ where $(')$ denotes the derivative.

Example Take the map derivative and the polynomials $g_i(x) = a_i x + b_i$. We have

$$g_1\partial g_2 = \{a_1 a_2 x + a_1 b_2, a_1 a_2 x + b_1 a_2\},$$

so this is a ∂ -hope. All $x+c$, c be constant, are units.

Properties 2.6 If $(G, \cdot)$ is a semigroup then: For every f , the (∂) is *WASS*, and the $(\underline{\partial})$ is *WASS*. If f is homomorphism and projection, then (∂) is associative.

Reproductivity. If $(\cdot)$ is reproductivity then (∂) is also reproductivity.

Commutativity. If $(\cdot)$ is commutative then (∂) is commutative. If f is into the centre of G , then (∂) is commutative. If $(\cdot)$ is *COW* then, (∂) is *COW*.

Unit elements. u is unit if $f(u) = e$, e unit of $(G, \cdot)$. The elements of the kernel of f , are ∂ -units.

Inverse elements. Let $(G, \cdot)$ monoid with unit e , u unit in (G, ∂) , so $f(u) = e$. The $x' = (f(x))^{-1}u$ and $x' = u(f(x))^{-1}$, are right and left inverses, resp. We have two-sided inverses iff $f(x)u = uf(x)$.

Proposition. Let $(G, \cdot)$ be a group then, $\forall f: G \rightarrow G$, the (G, ∂) is an H_v -group.

An important new application, which combines hyperstructure theory and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis [34]. The suggestion is the following:

Definition 2.7 In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:



The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point they feel expresses their answer to the specific question.

The use of the Vougiouklis & Vougiouklis bar instead of a Likert scale has several advantages during both the filling-in and the research processing. The final suggested length of the bar in printed form, after research and according to the Golden Ratio, is 6.2cm.

The well known class of P -hyperstructures based on classical ones are defined by [11], [16]:

Definition 2.8 Let $(G, \cdot)$ be groupoid, then for every $P \subset G$, $P \neq \emptyset$, we define the following hopes called P -hopes: $\forall x, y \in G$

$$\underline{P}: xPy = (xP)y \cup x(Py), \quad \underline{P}_r: xP_r y = (xy)P \cup x(yP), \quad \underline{P}_l: xP_l y = (Px)y \cup P(xy).$$

$(G, \underline{P})$, $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ are called P -hyperstructures. The most usual case is if $(G, \cdot)$ is semigroup, then

$$xPy = (xP)y \cup x(Py) = xPy$$

and $(G, \underline{P})$ is a semihypergroup

A generalization of P -hopes, used in Santilli's isothory, is the following [5], [9], [31]: Let $(G, \cdot)$ be abelian group and P a subset of G with $\#P > 1$. We define the hope $(\times_P)$ as follows:

$$x \times_P y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y \mid h \in P\} & \text{if } x \neq e \text{ and } y \neq e \\ x \cdot y & \text{if } x = e \text{ or } y = e \end{cases}$$

we call this hope P_e -hope. The hyperstructure $(G, \times_P)$ is abelian H_v -group.

New definitions on this topic appear [32], [33]:

Definition 2.9 Let $A = (a_{ij})$, $B = (b_{ij}) \in M_{m \times n}$, we call (A, B) unitize pair of matrices if $A^t B = I_n$, where I_n denotes the $n \times n$ unit matrix.

The following theorem can be applied in the classical theory, as well.

Theorem. If $m < n$, then there is no unitize pair.

Proof. Suppose that $n > m$ and that $A^t B = (c_{ij})$, $c_{ij} = \sum_{k=1}^m a_{ik} b_{kj}$. Denote by A_m the block of the matrix A such that $A_m = (a_{ij}) \in M_{m \times m}$, i.e. we take the matrix of the first m columns. Then we suppose that we have $(A_m)^t B_m = I_m$, therefore we must have $\det(A_m) \neq 0$. Now, since $n > m$, we can consider the homogeneous system with respect to the 'unknowns' $b_{1n}, b_{2n}, \dots, b_{mn}$:

$$c_{in} = \sum_{k=1}^m a_{ik} b_{kn} = 0 \quad \text{for } i = 1, 2, \dots, m.$$

From which, since $\det(A_m) \neq 0$, we obtain $b_{1n} = b_{2n} = \dots = b_{mn} = 0$. Using this on the last equation, on the same unknowns, $c_{nn} = \sum_{k=1}^m a_{nk} b_{kn} = 1$ we have $0 = 1$, absurd. ■

3. REPRESENTATIONS

Representations (we abbreviate by *rep*) of H_v -groups, can be considered either by H_v -matrices [12], [16], [17], [20] or by generalized permutations [15].

H_v -matrix (or *h/v-matrix*) is called a matrix with entries elements of an H_v -ring or H_v -field (or *h/v-ring* or *h/v-field*). The hyperproduct of H_v -matrices $A = (a_{ij})$ and $B = (b_{ij})$, of type $m \times n$ and $n \times r$, respectively, is a set of $m \times r$ H_v -matrices, defined in a usual manner:

$$A \cdot B = (a_{ij}) \cdot (b_{ij}) = \{C = (c_{ij}) \mid c_{ij} \in \oplus \Sigma a_{ik} \cdot b_{kj}\},$$

where $(\oplus)$ is the *n-ary circle hope* on the hypersum: the sum of products of elements of the H_v -ring is the union of the sets obtained with all possible parentheses on them. The hyperproduct of H_v -matrices is not necessarily *WASS*.

The rep problem by H_v -matrices is the following:

Definition 3.1 Let $(H, \cdot)$ be H_v -group, $(R, +, \cdot)$ be H_v -ring and $M_R = \{(a_{ij}) \mid a_{ij} \in R\}$, then any

$$T: H \rightarrow M_R: h \rightarrow T(h) \quad \text{with } T(h_1 h_2) \cap T(h_1)T(h_2) \neq \emptyset, \quad \forall h_1, h_2 \in H,$$

is called *H_v-matrix rep*. If $T(h_1h_2) \subset T(h_1)T(h_2)$, then T is an *inclusion rep*, if $T(h_1h_2) = T(h_1)T(h_2)$, then T is a *good rep* an induced rep T^* for the hypergroup algebra is obtained. If T is one to one and good then it is a *faithful rep*.

In reps of H_v -groups by H_v -matrices, there are two difficulties: To find an H_v -ring and an appropriate set of H_v -matrices. The problem of reps is very complicated because the cardinality of the product of two H_v -matrices is big. The problem can be simplified in several special cases such as the following:

- (a) The H_v -matrices are on H_v -rings with 0 and 1 and if these are scalars. Thus, the e-hyperstructures are interesting in the rep theory.
- (b) The H_v -matrices are over *very thin* H_v -rings.
- (c) The case of H_v -rings which contains singles, then these act as absorbings.

The main theorem of reps on H_v -structures [16], is the following:

Theorem 3.2 A necessary condition in order to have an inclusion rep T of an H_v -group $(H, \cdot)$ by $n \times n$ H_v -matrices over the H_v -ring $(R, +, \cdot)$ is the following: For all $\beta^*(x)$, $x \in H$ there must exist elements $a_{ij} \in H$, $i, j \in \{1, \dots, n\}$ such that

$$T(\beta^*(a)) \subset \{A = (a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}), i, j \in \{1, \dots, n\}\}$$

Thus, every inclusion rep $T: H \rightarrow M_R$: $a \mapsto T(a) = (a_{ij})$ induces a homomorphic rep T^* of H/β^* over R/γ^* by setting $T^*(\beta^*(a)) = [\gamma^*(a_{ij})]$, $\forall \beta^*(a) \in H/\beta^*$, where the element $\gamma^*(a_{ij}) \in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$. Then, T^* is called *fundamental induced rep* of T .

Denote $\text{tr}_\phi(T(x)) = \gamma^*(T(x_{ii}))$ the fundamental trace, then the mapping

$$X_T: H \rightarrow R/\gamma^*: x \mapsto X_T(x) = \text{tr}_\phi(T(x)) = \text{tr} T^*(x)$$

is called *fundamental character*. There are several types of traces.

Several constructions are used to obtain appropriate H_v -rings and H_v -fields:

- (i) Let $(H, \cdot)$ be H_v -group, then for every $(\oplus)$ such that $x \oplus y \supset \{x, y\}$, $\forall x, y \in H$, the $(H, \oplus, \cdot)$ is an H_v -ring. These H_v -rings are called *associated to* $(H, \cdot)$ H_v -rings. In rep theory of hypergroups, there are three associated hyperrings $(H, \oplus, \cdot)$ to $(H, \cdot)$. The $(\oplus)$ is defined respectively, $\forall x, y \in H$, as follows:

$$\text{type a: } x \oplus y = \{x, y\}, \text{ type b: } x \oplus y = \beta^*(x) \cup \beta^*(y), \text{ type c: } x \oplus y = H.$$

The strong associativity and strong or inclusion distributivity, is valid.

- (ii) Let $(H, +)$ be H_v -group, then for all hopes $(\otimes)$ such that $x \otimes y \supset \{x, y\}$, $\forall x, y \in H$, the $(H, +, \otimes)$ is an H_v -ring.

A variation is: Let 0 be a scalar, then $\forall(\otimes): x \otimes y \supset \{x, y\}, \forall x, y \in H - \{0\}, x \otimes 0 = 0 \otimes x = 0, \forall x \in H$, the $(H, +, \otimes)$ is an H_v -ring.

(iii) Let $(H, \cdot)$ be H_v -group. Take a $0 \notin H$ and set $H' = H \cup \{0\}$. Define a hope $(+)$ by: $0 + 0 = 0, 0 + x = H = x + 0, x + y = 0, \forall x, y \in H$, and extend $(\cdot)$ in H' by $0 \cdot 0 = 0, 0 \cdot x = x \cdot 0 = 0, \forall x, y \in H$. Then $(H', +, \cdot)$ is reproductive H_v -field with $H'/\gamma^* \cong Z_2$ where 0 is absorbing and single.

Hopes on any type of ordinary matrices can be defined [10], [35] they are called *helix hopes*.

Definition 3.3 Let $A = (a_{ij}) \in M_{m \times n}$ be matrix and $s, t \in \mathbb{N}$, with $1 \leq s \leq m, 1 \leq t \leq n$. The *helix-projection* is a map $\underline{st}: M_{m \times n} \rightarrow M_{s \times t}: A \rightarrow A \underline{st} = (\underline{a}_{ij})$, where $A \underline{st}$ has entries

$$\underline{a}_{ij} = \{a_{i+\kappa s, j+\lambda t} \mid 1 \leq i \leq s, 1 \leq j \leq t \text{ and } \kappa, \lambda \in \mathbb{N}, i + \kappa s \leq m, j + \lambda t \leq n\}$$

Let $A = (a_{ij}) \in M_{m \times n}, B = (b_{ij}) \in M_{u \times v}$ be matrices and $s = \min(m, u), t = \min(n, v)$. We define a hyper-addition, called *helix-sum*, by

$$\oplus : M_{m \times n} \times M_{u \times v} \rightarrow P(M_{s \times t}): (A, B) \rightarrow A \oplus B = A \underline{st} + B \underline{st} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset M_{s \times t}$$

where $(\underline{a}_{ij}) + (\underline{b}_{ij}) = \{(c_{ij}) = (a_{ij} + b_{ij}) \mid a_{ij} \in \underline{a}_{ij} \text{ and } b_{ij} \in \underline{b}_{ij}\}$.

Let $A = (a_{ij}) \in M_{m \times n}, B = (b_{ij}) \in M_{u \times v}$ and $s = \min(n, u)$. Define the *helix-product*, by

$$\otimes : M_{m \times n} \times M_{u \times v} \rightarrow P(M_{m \times v}): (A, B) \rightarrow A \otimes B = A \underline{ms} \cdot B \underline{sv} = (\underline{a}_{ij}) \cdot (\underline{b}_{ij}) \subset M_{m \times v}$$

where $(\underline{a}_{ij}) \cdot (\underline{b}_{ij}) = \{(c_{ij}) = (\sum a_{it} b_{tj}) \mid a_{ij} \in \underline{a}_{ij} \text{ and } b_{ij} \in \underline{b}_{ij}\}$.

The helix-sum is commutative, WASS, not associative. The helix-product is WASS, not associative and not distributive to the helix-addition.

Using several classes of H_v -structures one can face several representations. Some of those classes are as follows [5], [24], [26], [33]:

Definition 3.4 Let $M = M_{m \times n}$, the set of $m \times n$ matrices on R and $P = \{P_i; i \in I\} \subseteq M$. We define, a kind of, a P -hope $\underline{P}$ on M as follows

$$\underline{P}: M \times M \rightarrow P(M): (A, B) \underline{APB} = \{AP^t_i B; i \in I\} \subseteq M$$

where P^t denotes the transpose of P .

$\underline{P}$ is strong associative and inclusion distributive to addition:

$$\underline{AP}(B+C) \subseteq \underline{APB} + \underline{APC}, \forall A, B, C \in M$$

is valid. So $(M, +, \underline{P})$ defines a multiplicative hyperring on non-square matrices.

Definition 3.5 Let $M = M_{m \times n}$ be module of $m \times n$ matrices on R and take the sets

$$S = \{s_k : k \in K\} \subseteq R, \quad Q = \{Q_i : i \in J\} \subseteq M, \quad P = \{P_i : i \in I\} \subseteq M.$$

Define three hopes as follows

$$\begin{aligned} \underline{S} : R \times M &\rightarrow P(M) : (r, A) \rightarrow r \underline{S} A = \{(rs_k)A : k \in K\} \subseteq M \\ \underline{Q}_+ : M \times M &\rightarrow P(M) : (A, B) \rightarrow A \underline{Q}_+ B = \{A + Q_j + B : j \in J\} \subseteq M \\ \underline{P} : M \times M &\rightarrow P(M) : (A, B) \rightarrow A \underline{P} B = \{A P_i B : i \in I\} \subseteq M \end{aligned}$$

Then $(M, \underline{S}, \underline{Q}_+, \underline{P})$ is a hyperalgebra on R called *general matrix P-hyperalgebra*.

We present the rep problem is by *Generalized Permutations* (write *gp*) [15].

Definitions 3.6 Let X be a set, then a map $f : X \rightarrow P(X) - \{\emptyset\}$, is a *gp of X* if the reproduction axiom is valid

$$\bigcup_{x \in X} x f(x) = f(X) = X.$$

Denote by M_X the set of all gps on X . For an H_v -group $(X, \cdot)$ and $a \in X$, the gp f_a defined by $f_a(x) = ax$ is *inner gp*. Arrow of f is any $(x, y) \in X^2$ with $y \in f(x)$. $f_2 \in M_X$ contains $f_1 \in M_X$ or f_1 is a *sub-gp* of f_2 , if $f_1(x) \subset f_2(x)$, $\forall x \in X$, then we write $f_1 \subset f_2$. If, moreover, $f_1 \neq f_2$, then f_1 is a proper sub-gp of f_2 . A $f \in M_X$ is called *minimal* if it has no proper sub-gp. Denote $\underline{M}_X$ the set of all minimals. The gp t with $t(x) = X$, $\forall x \in X$, is called *universal*. The *converse* of f is the $\underline{f}$ defined by $\underline{f}(x) = \{z \in X : f(z) \ni x\}$, so, $\underline{f}$ has reversed arrows. We call *associated* to $f \in M_X$ the gp $f \circ \underline{f}$, where $(\circ)$ is the map composition.

The *Union* $f = \bigcup_{i \in I} f_i$ of a family of gps $\{f_i : i \in I\}$, is $f(x) = \bigcup_{i \in I} f_i(x)$, $\forall x \in X$.

Theorem 3.7 Let $f \in M_X$, then $f \in \underline{M}_X$ if and only if, the following condition is valid: if $a \neq b$ and $f(a) \cap f(b) \neq \emptyset$, then $f(a) = f(b)$ and $f(a)$ is a singleton.

Corollary. If $f \in \underline{M}_X$ then $\underline{f} \in \underline{M}_X$. Explicit description of

$$\underline{M}_X : (f \circ \underline{f})(x) = f\{u : f(u) \ni x\} = \bigcup_{f(u) \ni x} f(u), \quad \forall x \in X.$$

Thus, $(f \circ \underline{f})(x) = \{y : \exists u \in X, \{x, y\} \subset f(u)\}$. So, if I the identity, then $I \subset f \circ \underline{f}$, $\forall f \in M_X$.

Corollary. If $f \in \underline{M}_X$ then $(f \circ \underline{f})(x) = \{y \in X : f(y) = f(x)\}$.

4. THE SANTILLI'S ISO-THEORY. e-CONSTRUCTIONS

Santilli's theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting' of the n -dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n -dimensional new matrix. The original theory is

reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed correspond into the hyperstructures were introduced by Santilli & Vougiouklis in 1996, and they are called *e-hyperfields*. The H_v -fields can give *e-hyperfields* which can be used in the isotopy theory in applications as in physics or biology [4], [6], [7], [8], [9], [21], [25], [28], [29], [30], [31].

Definition 4.1 A hyperstructure $(H, \cdot)$ containing a unique scalar unit e and for all x , there exists an inverse x^{-1} , i.e. $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$, is called *e-hyperstructure*.

Definition 4.2 $(F, +, \cdot)$, where $(+)$ is operation and $(\cdot)$ hope, is called *e-hyperfield* if the following are valid: $(F, +)$ is an abelian group with the additive unit 0 ; $(\cdot)$ is WASS; $(\cdot)$ is weak distributive with respect to $(+)$; 0 is absorbing: $0 \cdot x = x \cdot 0 = 0$, $\forall x \in F$; there exist a multiplicative scalar unit 1 , i.e. $1 \cdot x = x \cdot 1 = x$, $\forall x \in F$, and $\forall x \in F$ there exists a unique inverse x^{-1} , such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.

The elements of an *e-hyperfield* are called *e-hypernumbers*. If the relation: $1 = x \cdot x^{-1} = x^{-1} \cdot x$, is valid, then we say that we have a *strong e-hyperfield*.

Definition 4.3 *The Main e-Construction*. Given a group $(G, \cdot)$, where e is the unit, then we define in G , a large number of hopes $(\otimes)$ as follows:

$$x \otimes y = \{xy, g_1, g_2, \dots\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, \dots \in G - \{e\}$$

$g_1, g_2, \dots$ are not the same for each pair (x, y) . $(G, \otimes)$ is H_v -group, it is H_b -group which contains the $(G, \cdot)$. $(G, \otimes)$ is *e-hypergroup*. Moreover, if for each x, y such that $xy = e$, we have $x \otimes y = xy$, then $(G, \otimes)$ becomes a strong *e-hypergroup*.

The proof is immediate since for both cases we enlarge the results of the group by putting elements from the set G and applying the Little Theorem. Moreover one can easily see that the unit e is a unique scalar element and for each x in G , there exists a unique inverse x^{-1} , such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$. Finally if the last condition is valid then we have $1 = x \cdot x^{-1} = x^{-1} \cdot x$, so the hyperstructure $(G, \otimes)$ is a strong *e-hypergroup*.

The main *e-construction* gives an extremely large number of *e-hopes*.

Example. Take the quaternion group $Q = \{1, -1, i, -i, j, -j, k, -k\}$ with defining relations $i^2 = j^2 = -1$, $ij = -ji = k$. Denoting $\underline{i} = \{i, -i\}$, $\underline{j} = \{j, -j\}$, $\underline{k} = \{k, -k\}$ we may define a very large number $(*)$ hopes by enlarging only few products. For example, take $(-1) * k = \underline{k}$, $k * i = \underline{j}$ and $i * j = \underline{k}$. Then, $(Q, *)$ is strong *e-hypergroup*.

According to Santilli's iso-theory [6], [7] on a field $F = (F, +, \cdot)$, a *general isofield* $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\cdot})$ is defined to be a field with elements $\hat{a} = a \cdot \hat{1}$, called

isonumbers, where $a \in F$, and $\hat{1}$ is a positive-defined element generally outside F , equipped with two operations $\hat{+}$ and $\hat{\times}$ where $\hat{+}$ is the sum with the conventional additive unit 0, and $\hat{\times}$ is a new multiplication

$$\hat{a} \hat{\times} \hat{b} := \hat{a} \times \hat{T} \times \hat{b}, \text{ with } \hat{1} = \hat{T}^{-1}, \forall \hat{a}, \hat{b} \in \hat{F} \quad (i)$$

called *iso-multiplication*, for which $\hat{1}$ is the left and right unit of $\hat{F}$,

$$\hat{1} \hat{\times} \hat{a} = \hat{a} \times \hat{1} = \hat{a}, \forall \hat{a} \in \hat{F} \quad (ii)$$

called *iso-unit*. The rest properties of a field are reformulated analogously.

In order to transfer this theory into the hyperstructure case we generalize only the new multiplication $\hat{\times}$ from (i), by replacing with a hope including the old one. We introduce two general constructions on this direction as follows:

Construction 4.4 General enlargement. On a field $F=(F,+, \cdot)$ and the isofield $\hat{F}=\hat{F}(\hat{a}, \hat{+}, \hat{\times})$ we replace in the results of the iso-product

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \times \hat{T} \times \hat{b}, \text{ with } \hat{1} = \hat{T}^{-1}$$

of the element $\hat{T}$ by a set of elements $\hat{H}_{ab}=\{\hat{T}, \hat{x}_1, \hat{x}_2, \dots\}$ where $\hat{x}_1, \hat{x}_2, \dots \in \hat{F}$, containing $\hat{T}$, for all $\hat{a} \hat{\times} \hat{b}$ for which $\hat{a}, \hat{b} \notin \{\hat{0}, \hat{1}\}$ and $\hat{x}_1, \hat{x}_2, \dots \in \hat{F} - \{\hat{0}, \hat{1}\}$. If one of $\hat{a}$, $\hat{b}$, or both, is equal to $\hat{0}$ or $\hat{1}$, then $\hat{H}_{ab}=\{\hat{T}\}$. Therefore the new iso-hope is

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \times \hat{H}_{ab} \times \hat{b} = \hat{a} \times \{\hat{T}, \hat{x}_1, \hat{x}_2, \dots\} \times \hat{b}, \forall \hat{a}, \hat{b} \in \hat{F} \quad (iii)$$

$\hat{F}=\hat{F}(\hat{a}, \hat{+}, \hat{\times})$ becomes *isoH_v-field*. The elements of F are called *isoH_v-numbers*.

Remarks. More important hopes, of the above construction, are the ones where only for few ordered pairs $(\hat{a}, \hat{b})$ the result is enlarged, even more, the extra elements $\hat{x}_i$, are only few, preferable exactly one. Thus, this special case is if there exists only one pair $(\hat{a}, \hat{b})$ for which

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \times \{\hat{T}, \hat{x}\} \times \hat{b}, \forall \hat{a}, \hat{b} \in \hat{F}$$

and the rest are ordinary results, then we have a *very thin isoH_v-field*.

The assumption that $\hat{H}_{ab}=\{\hat{T}\}$, $\hat{a}$ or $\hat{b}$, is equal to $\hat{0}$ or $\hat{1}$, with that $\hat{x}_i$, are not $\hat{0}$ or $\hat{1}$, give that the isoH_v-field has one scalar absorbing $\hat{0}$, one scalar $\hat{1}$, and $\forall \hat{a} \in \hat{F}$, has one inverse.

Construction 4.3 The P-hope. Consider an isofield $\hat{F}=\hat{F}(\hat{a}, \hat{+}, \hat{\times})$ with $\hat{a}=a \times \hat{1}$, the isonumbers, where $a \in F$, and $\hat{1}$ is a positive-defined element, outside F ,

with two operations $\hat{+}$ and $\hat{\times}$, where $\hat{+}$ is the sum with the conventional unit 0, and $\hat{\times}$ is the iso-multiplication

$$\hat{a} \hat{\times} \hat{b} := \hat{a} \times \hat{T} \times \hat{b}, \text{ with } \hat{1} = \hat{T}^{-1}, \forall \hat{a}, \hat{b} \in \hat{F}$$

Take a set $\hat{P} = \{\hat{T}, \hat{p}_1, \dots, \hat{p}_s\}$, with $\hat{p}_1, \dots, \hat{p}_s \in \hat{F} - \{\hat{0}, \hat{1}\}$, define the *isoP-H_v-field*, $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times}_P)$, where the hope $\hat{\times}_P$ as follows:

$$\hat{a} \hat{\times}_P \hat{b} := \begin{cases} \hat{a} \times \hat{P} \times \hat{b} = \{\hat{a} \times \hat{h} \times \hat{b} \mid \hat{h} \in \hat{P}\} & \text{if } \hat{a} \neq \hat{1} \text{ and } \hat{b} \neq \hat{1} \\ \hat{a} \times \hat{T} \times \hat{b} & \text{if } \hat{a} = \hat{1} \text{ or } \hat{b} = \hat{1} \end{cases} \quad (\text{iv})$$

The elements of $\hat{F}$ are called *isoP-H_v-numbers*.

Remark. If $\hat{P} = \{\hat{T}, \hat{p}\}$, that is that $\hat{P}$ contains only one $\hat{p}$ except $\hat{T}$. The inverses in isoP-H_v-fields, are not necessarily unique.

5. SMALL HYPERNUMBERS

‘What are the hypernumbers?’ this is the basic question on the Santilli’s isothory extended on hyperstructures. Since the H_v-structures are the largest class of hyperstructures, we present a way to reach stricter hyperstructures.

We are interested for the small multiplicative h/v-fields defined on $(\mathbb{Z}_n, +, \cdot)$, which satisfy the following conditions:

- (a) they have non-degenerate fundamental field,
- (b) they are very thin minimal,
- (c) they are COW (non-commutative), and
- (d) they have 0 and 1, scalars.

Theorem 5.1 On the small ring $(\mathbb{Z}_4, +, \cdot)$ all the multiplicative h/v-fields which have the above conditions are the following:

We have the isomorphic cases: $2 \otimes 3 = \{0, 2\}$ or $3 \otimes 2 = \{0, 2\}$. The fundamental classes are $[0] = \{0, 2\}$, $[1] = \{1, 3\}$ and we have

$$(\mathbb{Z}_4, +, \otimes) / \gamma^* \cong (\mathbb{Z}_2, +, \cdot).$$

Thus it is isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$. In this H_v-group there is only one unit and every element has a unique double inverse.

Theorem 5.2 All multiplicative h/v-fields on $(\mathbb{Z}_6, +, \cdot)$, which satisfy the above conditions, are the following isomorphic cases:

We have the only one hyperproduct,

- (I) $2 \otimes 3 = \{0, 3\}$ or $2 \otimes 4 = \{2, 5\}$
 $3 \otimes 4 = \{0, 3\}$ or $3 \otimes 5 = \{0, 3\}$ or $4 \otimes 5 = \{2, 5\}$

Fundamental classes: $[0] = \{0, 3\}$, $[1] = \{1, 4\}$, $[2] = \{2, 5\}$ and

$$(\mathbb{Z}_6, +, \otimes) / \gamma^* \cong (\mathbb{Z}_3, +, \cdot).$$

- (II) $2 \otimes 3 = \{0, 2\}$ or $2 \otimes 3 = \{0, 4\}$ or $2 \otimes 4 = \{0, 2\}$ or $2 \otimes 4 = \{2, 4\}$ or
 $2 \otimes 5 = \{0, 4\}$ or $2 \otimes 5 = \{2, 4\}$ or $3 \otimes 4 = \{0, 2\}$ or $3 \otimes 4 = \{0, 4\}$ or
 $3 \otimes 5 = \{3, 5\}$ or $4 \otimes 5 = \{0, 2\}$ or $4 \otimes 5 = \{2, 4\}$.

Fundamental classes: $[0] = \{0, 2, 4\}$, $[1] = \{1, 3, 5\}$ and

$$(\mathbb{Z}_6, +, \otimes) / \gamma^* \cong (\mathbb{Z}_2, +, \cdot).$$

Remark that if we need h/v-fields where the elements have at most one inverse element, then we must exclude the case of $2 \otimes 5 = \{1, 4\}$ from (I), and the case $3 \otimes 5 = \{1, 3\}$ from (II).

Example 5.3 In h/v-field $(\mathbb{Z}_6, +, \otimes)$ where the only hyperproduct is $2 \otimes 4 = \{2, 5\}$ we consider the h/v-matrices of type

$$\underline{i} = E_{11} + iE_{12} + 4E_{22}, \text{ where } i=0, 1, 2, 3, 4, 5,$$

then the multiplicative table of the hyperproduct of h/v-matrices is given by

$\otimes$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	4	5	0	1	2	3
2	2	0, 3	1, 4	2, 5	0, 3	1, 4
3	0	1	2	3	4	5
4	4	5	0	1	2	3
5	2	3	4	5	0	1

where the fundamental classes are $(0) = \{0, 3\}$, $(1) = \{1, 4\}$, $(2) = \{2, 5\}$ and the fundamental group is isomorphic to $(\mathbb{Z}_3, +)$. Remark that $(\mathbb{Z}_6, \otimes)$ is an h/v-group which is cyclic where the elements $\underline{2}$ and $\underline{4}$ are generators of period 4.

We conclude with a general theorem on h/v-fields of type $(\mathbb{Z}, +, \otimes)$,

Theorem 5.4 A construction of a large number of multiplicative h/v-fields on $(\mathbb{Z}_{pq}, +, \cdot)$, where p, q are prime numbers, which satisfy the conditions: (a) have non-degenerate fundamental field, (b) are very thin minimal, (c) are COW (non-commutative), and (d) have 0 and 1, scalars, can be defined as follows:

We have the only one hyperproduct which can be of the type

$$2 \otimes 3 = \{k, p+k\}, \text{ for } k=0, 2, 3, \dots, q-1.$$

Then the fundamental classes are:

$$[0] = \{0, p\}, [1] = \{1, p+1\}, [q-1] = \{q-1, q+p-1\}$$

and we have

$$(\mathbb{Z}_{pq}, +, \otimes) / \gamma^* \cong (\mathbb{Z}_q, +, \cdot).$$

Proof. Straightforward.

Remark that in this theorem we can change the role of p and q and then we obtain the analogous result. Moreover, this not the only case to have the same result, therefore the construction is more general.

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COMPLEX HYPERBOLIC FUNCTIONS (I)
AND FERMAT'S LAST THEOREM

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Abstract

In 1637 Fermat wrote: "*It is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates, or in general any power higher than the second into powers of like degree: I have discovered a proof, which this margin is too small to contain.*" This means:

$x^n + y^n = z^n$ ($n > 2$) has no integer solution, all different from 0 (i.e., it has only the trivial solution, where one of the integers is equal to 0). It has been called Fermat's last theorem (FLT). It suffices to prove FLT for exponent 4, and every prime exponent P . Fermat proved FLT for exponent 4. Euler proved FLT for exponent 3.

Key words complex hyperbolic, Euler proof.

EDITORIAL NOTE:

The submission of articles critical of this paper would be greatly appreciated for publication in Algebras, Groups and Geometries.

In this paper using the complex hyperbolic functions we prove FLT for exponents $3P$ and P , where P is an odd prime. The proof of FLT must be direct. But indirect proof of FLT is disbelieving.

In 1974 Jiang found out Euler formula of the cyclotomic real numbers in the cyclotomic fields

$$\exp\left(\sum_{i=1}^{n-1} t_i J^i\right) = \sum_{i=1}^n S_i J^{i-1} \quad (1)$$

where J denotes a n th root of unity, $J^n = 1$, n is an odd number, t_i are the real numbers.

S_i is called the complex hyperbolic functions of order n with $n-1$ variables [1-7].

$$S_i = \frac{1}{n} \left[e^A + 2 \sum_{j=1}^{\frac{n-1}{2}} (-1)^{(i-1)j} e^{B_j} \cos\left(\theta_j + (-1)^j \frac{(i-1)j\pi}{n}\right) \right] \quad (2)$$

where $i=1,2,\dots,n$;

$$\begin{aligned} A &= \sum_{\alpha=1}^{n-1} t_{\alpha}, & B_j &= \sum_{\alpha=1}^{n-1} t_{\alpha} (-1)^{\alpha j} \cos \frac{\alpha j \pi}{n}, \\ \theta_j &= (-1)^{j+1} \sum_{\alpha=1}^{n-1} t_{\alpha} (-1)^{\alpha j} \sin \frac{\alpha j \pi}{n}, & A + 2 \sum_{j=1}^{\frac{n-1}{2}} B_j &= 0 \end{aligned} \quad (3)$$

(2) may be written in the matrix form

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_n \end{bmatrix} = \frac{1}{n} \begin{bmatrix} 1 & 1 & 0 & \vdots & 0 \\ 1 & -\cos \frac{\pi}{n} & -\sin \frac{\pi}{n} & \vdots & -\sin \frac{(n-1)\pi}{2n} \\ 1 & \cos \frac{2\pi}{n} & \sin \frac{2\pi}{n} & \vdots & -\sin \frac{(n-1)\pi}{n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cos \frac{(n-1)\pi}{n} & \sin \frac{(n-1)\pi}{n} & \vdots & -\sin \frac{(n-1)^2\pi}{2n} \end{bmatrix} \begin{bmatrix} e^A \\ 2e^{B_1} \cos \theta_1 \\ 2e^{B_1} \sin \theta_1 \\ \vdots \\ 2 \exp B_{\frac{n-1}{2}} \sin \theta_{\frac{n-1}{2}} \end{bmatrix} \quad (4)$$

where $(n-1)/2$ is an even number.

From (4) we have its inverse transformation

$$\begin{bmatrix} e^A \\ e^{B_1} \cos \theta_1 \\ e^{B_1} \sin \theta_1 \\ L \\ \exp(B_{\frac{n-1}{2}}) \sin(\theta_{\frac{n-1}{2}}) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & L & 1 \\ 1 & -\cos \frac{\pi}{n} & \cos \frac{2\pi}{n} & L & \cos \frac{(n-1)\pi}{n} \\ 0 & -\sin \frac{\pi}{n} & \sin \frac{2\pi}{n} & L & \sin \frac{(n-1)\pi}{n} \\ L & L & L & L & L \\ 0 & -\sin \frac{(n-1)\pi}{2n} & -\sin \frac{(n-1)\pi}{n} & L & -\sin \frac{(n-1)^2\pi}{2n} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ L \\ S_n \end{bmatrix} \quad (5)$$

From (5) we have

$$\begin{aligned} e^A &= \sum_{i=1}^n S_i, \quad e^{B_j} \cos \theta_j = S_1 + \sum_{i=1}^{n-1} S_{1+i} (-1)^j \cos \frac{ij\pi}{n} \\ e^{B_j} \sin \theta_j &= (-1)^{j+1} \sum_{i=1}^{n-1} S_{1+i} (-1)^j \sin \frac{ij\pi}{n}, \end{aligned} \quad (6)$$

In (3) and (6) t_i and S_i have the same formulas. (4) and (5) are the most critical formulas of proofs for FLT. Using (4) and (5) in 1991 Jiang invented that every factor of exponent n has the Fermat equation and proved FLT [1-7] Substituting (4) into (5) we prove (5).

$$\begin{bmatrix} e^A \\ e^{B_1} \cos \theta_1 \\ e^{B_1} \sin \theta_1 \\ L \\ \exp(B_{\frac{n-1}{2}}) \sin(\theta_{\frac{n-1}{2}}) \end{bmatrix} = \frac{1}{n} \begin{bmatrix} 1 & 1 & 1 & L & 1 \\ 1 & -\cos \frac{\pi}{n} & \cos \frac{2\pi}{n} & L & \cos \frac{(n-1)\pi}{n} \\ 0 & -\sin \frac{\pi}{n} & \sin \frac{2\pi}{n} & L & \sin \frac{(n-1)\pi}{n} \\ L & L & L & L & L \\ 0 & -\sin \frac{(n-1)\pi}{2n} & -\sin \frac{(n-1)\pi}{n} & L & -\sin \frac{(n-1)^2\pi}{2n} \end{bmatrix} \times$$

$$\begin{aligned}
 & \begin{bmatrix} 1 & 1 & 0 & L & 0 \\ 1 & -\cos \frac{\pi}{n} & -\sin \frac{\pi}{n} & L & -\sin \frac{(n-1)\pi}{2n} \\ 1 & \cos \frac{2\pi}{n} & \sin \frac{2\pi}{n} & L & -\sin \frac{(n-1)\pi}{n} \\ L & L & L & L & L \\ 1 & \cos \frac{(n-1)\pi}{n} & \sin \frac{(n-1)\pi}{n} & L & -\sin \frac{(n-1)^2\pi}{2n} \end{bmatrix} \begin{bmatrix} e^A \\ 2e^{B_1} \cos \theta_1 \\ 2e^{B_1} \sin \theta_1 \\ L \\ 2 \exp(B_{\frac{n-1}{2}}) \sin(\theta_{\frac{n-1}{2}}) \end{bmatrix} \\
 &= \frac{1}{n} \begin{bmatrix} n & 0 & 0 & L & 0 \\ 0 & \frac{n}{2} & 0 & L & 0 \\ 0 & 0 & \frac{n}{2} & L & 0 \\ K & L & L & L & L \\ 0 & 0 & 0 & L & \frac{n}{2} \end{bmatrix} \begin{bmatrix} e^A \\ 2e^{B_1} \cos \theta_1 \\ 2e^{B_1} \sin \theta_1 \\ L \\ 2 \exp(B_{\frac{n-1}{2}}) \sin(\theta_{\frac{n-1}{2}}) \end{bmatrix} \\
 &= \begin{bmatrix} e^A \\ e^{B_1} \cos \theta_1 \\ e^{B_1} \sin \theta_1 \\ L \\ \exp(B_{\frac{n-1}{2}}) \sin(\theta_{\frac{n-1}{2}}) \end{bmatrix}, \tag{7}
 \end{aligned}$$

where $1 + \sum_{j=1}^{n-1} (\cos \frac{j\pi}{n})^2 = \frac{n}{2}$, $\sum_{j=1}^{n-1} (\sin \frac{j\pi}{n})^2 = \frac{n}{2}$.

From (3) we have

$$\exp(A + 2 \sum_{j=1}^{\frac{n-1}{2}} B_j) = 1. \tag{8}$$

From (6) we have

$$\exp(A + 2 \sum_{j=1}^{\frac{n-1}{2}} B_j) = \begin{vmatrix} S_1 & S_n & L & S_2 \\ S_2 & S_1 & L & S_3 \\ L & L & L & L \\ S_n & S_{n-1} & L & S_1 \end{vmatrix} = \begin{vmatrix} S_1 & (S_1)_1 & L & (S_1)_{n-1} \\ S_2 & (S_2)_1 & L & (S_2)_{n-1} \\ L & L & L & L \\ S_n & (S_n)_1 & L & (S_n)_{n-1} \end{vmatrix}, \quad (9)$$

where $(S_i)_j = \frac{\partial S_i}{\partial t_j} [7].$

From (8) and (9) we have the circulant determinant

$$\exp(A + 2 \sum_{j=1}^{\frac{n-1}{2}} B_j) = \begin{vmatrix} S_1 & S_n & L & S_2 \\ S_2 & S_1 & L & S_3 \\ L & L & L & L \\ S_n & S_{n-1} & L & S_1 \end{vmatrix} = 1 \quad (10)$$

If $S_i \neq 0$, where $i = 1, 2, L, n$, then (10) has infinitely many rational solutions.

Assume $S_1 \neq 0$, $S_2 \neq 0$, $S_i = 0$ where $i = 3, 4, L, n$. $S_i = 0$ are $n-2$ indeterminate equations with $n-1$ variables. From (6) we have

$$e^A = S_1 + S_2, \quad e^{2B_j} = S_1^2 + S_2^2 + 2S_1S_2(-1)^j \cos \frac{j\pi}{n}. \quad (11)$$

From (10) and (11) we have the Fermat equation

$$\exp(A + 2 \sum_{j=1}^{\frac{n-1}{2}} B_j) = (S_1 + S_2) \prod_{j=1}^{\frac{n-1}{2}} (S_1^2 + S_2^2 + 2S_1S_2(-1)^j \cos \frac{j\pi}{n}) = S_1^n + S_2^n = 1 \quad (12)$$

Example[1]. Let $n = 15$. From (3) we have

$$A = (t_1 + t_{14}) + (t_2 + t_{13}) + (t_3 + t_{12}) + (t_4 + t_{11}) + (t_5 + t_{10}) + (t_6 + t_9) + (t_7 + t_8)$$

$$B_1 = -(t_1 + t_{14}) \cos \frac{\pi}{15} + (t_2 + t_{13}) \cos \frac{2\pi}{15} - (t_3 + t_{12}) \cos \frac{3\pi}{15} + (t_4 + t_{11}) \cos \frac{4\pi}{15} \\ - (t_5 + t_{10}) \cos \frac{5\pi}{15} + (t_6 + t_9) \cos \frac{6\pi}{15} - (t_7 + t_8) \cos \frac{7\pi}{15},$$

$$\begin{aligned}
 B_2 &= (t_1 + t_{14}) \cos \frac{2\pi}{15} + (t_2 + t_{13}) \cos \frac{4\pi}{15} + (t_3 + t_{12}) \cos \frac{6\pi}{15} + (t_4 + t_{11}) \cos \frac{8\pi}{15} \\
 &\quad + (t_5 + t_{10}) \cos \frac{10\pi}{15} + (t_6 + t_9) \cos \frac{12\pi}{15} + (t_7 + t_8) \cos \frac{14\pi}{15}, \\
 B_3 &= -(t_1 + t_{14}) \cos \frac{3\pi}{15} + (t_2 + t_{13}) \cos \frac{6\pi}{15} - (t_3 + t_{12}) \cos \frac{9\pi}{15} + (t_4 + t_{11}) \cos \frac{12\pi}{15} \\
 &\quad - (t_5 + t_{10}) \cos \frac{15\pi}{15} + (t_6 + t_9) \cos \frac{18\pi}{15} - (t_7 + t_8) \cos \frac{21\pi}{15}, \\
 B_4 &= (t_1 + t_{14}) \cos \frac{4\pi}{15} + (t_2 + t_{13}) \cos \frac{8\pi}{15} + (t_3 + t_{12}) \cos \frac{12\pi}{15} + (t_4 + t_{11}) \cos \frac{16\pi}{15} \\
 &\quad + (t_5 + t_{10}) \cos \frac{20\pi}{15} + (t_6 + t_9) \cos \frac{24\pi}{15} + (t_7 + t_8) \cos \frac{28\pi}{15}, \\
 B_5 &= -(t_1 + t_{14}) \cos \frac{5\pi}{15} + (t_2 + t_{13}) \cos \frac{10\pi}{15} - (t_3 + t_{12}) \cos \frac{15\pi}{15} + (t_4 + t_{11}) \cos \frac{20\pi}{15} \\
 &\quad - (t_5 + t_{10}) \cos \frac{25\pi}{15} + (t_6 + t_9) \cos \frac{30\pi}{15} - (t_7 + t_8) \cos \frac{35\pi}{15}, \\
 B_6 &= (t_1 + t_{14}) \cos \frac{6\pi}{15} + (t_2 + t_{13}) \cos \frac{12\pi}{15} + (t_3 + t_{12}) \cos \frac{18\pi}{15} + (t_4 + t_{11}) \cos \frac{24\pi}{15} \\
 &\quad + (t_5 + t_{10}) \cos \frac{30\pi}{15} + (t_6 + t_9) \cos \frac{36\pi}{15} + (t_7 + t_8) \cos \frac{42\pi}{15}, \\
 B_7 &= -(t_1 + t_{14}) \cos \frac{7\pi}{15} + (t_2 + t_{13}) \cos \frac{14\pi}{15} - (t_3 + t_{12}) \cos \frac{21\pi}{15} + (t_4 + t_{11}) \cos \frac{28\pi}{15} \\
 &\quad - (t_5 + t_{10}) \cos \frac{35\pi}{15} + (t_6 + t_9) \cos \frac{42\pi}{15} - (t_7 + t_8) \cos \frac{49\pi}{15}, \\
 A + 2 \sum_{j=1}^7 B_j &= 0, \quad A + 2B_3 + 2B_6 = 5(t_5 + t_{10}). \tag{13}
 \end{aligned}$$

Form (12) we have the Fermat equation

$$\exp(A + 2 \sum_{j=1}^7 B_j) = S_1^{15} + S_2^{15} = (S_1^5)^3 + (S_2^5)^3 = 1. \tag{14}$$

From (13) we have

$$\exp(A + 2B_3 + 2B_6) = [\exp(t_5 + t_{10})]^5. \tag{15}$$

From (11) we have

$$\exp(A + 2B_3 + 2B_6) = S_1^5 + S_2^5. \quad (16)$$

From (15) and (16) we have the Fermat equation

$$\exp(A + 2B_3 + 2B_6) = S_1^5 + S_2^5 = [\exp(t_5 + t_{10})]^5. \quad (17)$$

Euler proved that (14) has no rational solutions for exponent 3[8]. Therefore we prove that (17) has no rational solutions for exponent 5[1].

Theorem 1. [1-7]. Let $n = 3P$, where $P > 3$ is odd prime. From (12) we have the Fermat's equation

$$\exp(A + 2 \sum_{j=1}^{\frac{3P-1}{2}} B_j) = S_1^{3P} + S_2^{3P} = (S_1^P)^3 + (S_2^P)^3 = 1. \quad (18)$$

From (3) we have

$$\exp(A + 2 \sum_{j=1}^{\frac{P-1}{2}} B_{3j}) = [\exp(t_p + t_{2p})]^P. \quad (19)$$

From (11) we have

$$\exp(A + 2 \sum_{j=1}^{\frac{P-1}{2}} B_{3j}) = S_1^P + S_2^P. \quad (20)$$

From (19) and (20) we have the Fermat equation

$$\exp(A + 2 \sum_{j=1}^{\frac{P-1}{2}} B_{3j}) = S_1^P + S_2^P = [\exp(t_p + t_{2p})]^P. \quad (21)$$

Euler proved that (18) has no rational solutions for exponent 3[8]. Therefore we prove that (21) has no rational solutions for $P > 3$ [1, 3-7].

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**COMPLEX HYPERBOLIC FUNCTIONS (II)
AND FERMAT'S LAST THEOREM**

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Abstract

In the preceding paper I, we presented a new proof of Fermat Last Theorem (FLT). In this paper using the complex hyperbolic functions we prove FLT for exponents $6P$ and P , where P is an odd prime. The proof of FLT must be direct. But indirect proof of FLT is disbelieving.

Key words complex hyperbolic, Euler proof.

EDITORIAL NOTE:

The submission of articles critical of this paper would be greatly appreciated for publication in Algebras, Groups and Geometries.

In 1974 Jiang found out Euler formula of the cyclotomic real numbers in the cyclotomic fields

$$\exp\left(\sum_{i=1}^{2n-1} t_i J^i\right) = \sum_{i=1}^{2n} S_i J^{i-1} \quad (1)$$

where J denotes a $2n$ th root of unity, $J^{2n} = 1$, n is an odd number, t_i are the real numbers.

S_i is called the complex hyperbolic functions of order $2n$ with $2n-1$ variables [5,7].

$$\begin{aligned} S_i = & \frac{1}{2n} \left[e^{A_i} + 2 \sum_{j=1}^{\frac{n-1}{2}} (-1)^{(i-1)jB_j} \cos\left(\theta_j + (-1)^j \frac{(i-1)j\pi}{n}\right) \right] \\ & + \frac{(-1)^{(i-1)}}{2n} \left[e^{A_2} + 2 \sum_{j=1}^{\frac{n-1}{2}} (-1)^{(i-1)j} e^{D_j} \cos\left(\phi_j + (-1)^{j+1} \frac{(i-1)j\pi}{n}\right) \right], \end{aligned} \quad (2)$$

where $i = 1, \dots, 2n$;

$$\begin{aligned} A_1 &= \sum_{\alpha=1}^{2n-1} t_\alpha, \quad B_j = \sum_{\alpha=1}^{2n-1} t_\alpha (-1)^{\alpha j} \cos \frac{\alpha j \pi}{n}, \quad \theta_j = (-1)^{(j+1)} \sum_{\alpha=1}^{2n-1} t_\alpha (-1)^{\alpha j} \sin \frac{\alpha j \pi}{n}, \\ A_2 &= \sum_{\alpha=1}^{2n-1} t_\alpha (-1)^\alpha, \quad D_j = \sum_{\alpha=1}^{2n-1} t_\alpha (-1)^{(j-1)\alpha} \cos \frac{\alpha j \pi}{n}, \\ \phi_j &= (-1)^j \sum_{\alpha=1}^{2n-1} t_\alpha (-1)^{(j-1)\alpha} \sin \frac{\alpha j \pi}{n}, \quad A_1 + A_2 + 2 \sum_{j=1}^{\frac{n-1}{2}} (B_j + D_j) = 0 \end{aligned} \quad (3)$$

From (2) we have its inverse transformation[5,7]

$$\begin{aligned} e^{A_i} &= \sum_{i=1}^{2n} S_i, \quad e^{A_2} = \sum_{i=1}^{2n} S_i (-1)^{1+i} \\ e^{B_j} \cos \theta_j &= S_1 + \sum_{i=1}^{2n-1} S_{1+i} (-1)^{ij} \cos \frac{ij\pi}{n}, \\ e^{B_j} \sin \theta_j &= (-1)^{(j+1)} \sum_{i=1}^{2n-1} S_{1+i} (-1)^{ij} \sin \frac{ij\pi}{n}, \end{aligned}$$

$$e^{D_j} \cos \phi_j = S_1 + \sum_{i=1}^{2n-1} S_{1+i} (-1)^{(j-1)i} \cos \frac{ij\pi}{n}$$

$$e^{D_j} \sin \phi_j = (-1)^j \sum_{i=1}^{2n-1} S_{1+i} (-1)^{(j-1)i} \sin \frac{ij\pi}{n} \quad (4)$$

(3) and (4) have the same form.

From (3) we have

$$\exp \left[A_1 + A_2 + 2 \sum_{j=1}^{\frac{n-1}{2}} (B_j + D_j) \right] = 1 \quad (5)$$

From (4) we have

$$\exp \left[A_1 + A_2 + 2 \sum_{j=1}^{\frac{n-1}{2}} (B_j + D_j) \right] = \begin{vmatrix} S_1 & S_{2n} & L & S_2 \\ S_2 & S_1 & L & S_3 \\ L & L & L & L \\ S_{2n} & S_{2n-1} & L & S_1 \end{vmatrix}$$

$$= \begin{vmatrix} S_1 & (S_1)_1 & L & (S_1)_{2n-1} \\ S_2 & (S_2)_1 & L & (S_2)_{2n-1} \\ L & L & L & L \\ S_{2n} & (S_{2n})_1 & L & (S_{2n})_{2n-1} \end{vmatrix} \quad (6)$$

where $(S_i)_j = \frac{\partial S_i}{\partial t_j} [7]$.

From (5) and (6) we have circulant determinant

$$\exp \left[A_1 + A_2 + 2 \sum_{j=1}^{\frac{n-1}{2}} (B_j + D_j) \right] = \begin{vmatrix} S_1 & S_{2n} & L & S_2 \\ S_2 & S_1 & L & S_3 \\ L & L & L & L \\ S_{2n} & S_{2n-1} & L & S_1 \end{vmatrix} = 1 \quad (7)$$

If $S_i \neq 0$, where $i = 1, 2, 3, \dots, 2n$, then (7) have infinitely many rational solutions.

Let $n = 1$. From (3) we have $A_1 = t_1$ and $A_2 = -t_1$. From (2) we have

$$S_1 = \text{ch } t_1 \quad S_2 = \text{sh } t_1 \quad (8)$$

we have Pythagorean theorem

$$\text{ch}^2 t_1 - \text{sh}^2 t_1 = 1 \quad (9)$$

(9) has infinitely many rational solutions.

Assume $S_1 \neq 0, S_2 \neq 0, S_i \neq 0$, where $i = 3, \dots, 2n$. $S_i = 0$ are $(2n-2)$ indeterminate equations with $(2n-1)$ variables. From (4) we have

$$\begin{aligned} e^{A_1} &= S_1 + S_2, \quad e^{A_2} = S_1 - S_2, \quad e^{2B_j} = S_1^2 + S_2^2 + 2S_1 S_2 (-1)^j \cos \frac{j\pi}{n}, \\ e^{2D_j} &= S_1^2 + S_2^2 + 2S_1 S_2 (-1)^{j+1} \cos \frac{j\pi}{n} \end{aligned} \quad (10)$$

Example. Let $n = 15$. From (3) and (10) we have Fermat's equation

$$\exp[A_1 + A_2 + 2 \sum_{j=1}^7 (B_j + D_j)] = S_1^{30} - S_2^{30} = (S_1^{10})^3 - (S_2^{10})^3 = 1 \quad (11)$$

From (3) we have

$$\exp(A_1 + 2B_3 + 2B_6) = [\exp(\sum_{j=1}^5 t_{5j})]^5 \quad (12)$$

From (10) we have

$$\exp(A_1 + 2B_3 + 2B_6) = S_1^5 + S_2^5 \quad (13)$$

From (12) and (13) we have Fermat's equation

$$\exp(A_1 + 2B_3 + 2B_6) = S_1^5 + S_2^5 = [\exp(\sum_{j=1}^5 t_{5j})]^5 \quad (14)$$

Euler prove that (11) has no rational solutions for exponent 3 [8]. Therefore we prove that (14) has no rational solutions for exponent 5.

Theorem. Let $n = 3P$ where P is an odd prime. From (7) and (8) we have Fermat's equation

$$\exp(A_1 + A_2 + 2 \sum_{j=1}^{\frac{3P-1}{2}} (B_j + D_j)) = S_1^{6P} - S_2^{6P} = (S_1^{2P})^3 - (S_2^{2P})^3 = 1 \quad (15)$$

From (3) we have

$$\exp\left(A_1 + 2 \sum_{j=1}^{\frac{P-1}{2}} B_{3j}\right) = \left[\exp\left(\sum_{j=1}^s t_{jP}\right)\right]^P \quad (16)$$

From (10) we have

$$\exp\left(A_1 + 2 \sum_{j=1}^{\frac{P-1}{2}} B_{3j}\right) = S_1^P + S_2^P \quad (17)$$

From (16) and (17) we have Fermat's equation

$$\exp\left(A_1 + 2 \sum_{j=1}^{\frac{P-1}{2}} B_{3j}\right) = S_1^P + S_2^P = \left[\exp\left(\sum_{j=1}^s t_{jP}\right)\right]^P \quad (18)$$

Euler prove that (15) has no rational solutions for exponent 3[8]. Therefore we prove that (18) has no rational solutions for prime exponent P [5,7].

Remark. It suffices to prove FLT for exponent 4. Let $n = 4P$, where P is an odd prime. We have the Fermat's equation for exponent $4P$ and the Fermat's equation for exponent P [2,5,7]. This is the proof that Fermat thought to have had. In complex hyperbolic functions let exponent n be $n = \Pi P$, $n = 2\Pi P$ and $n = 4\Pi P$. Every factor of exponent n has the Fermat's equation [1-7]. In complex trigonometric functions let exponent n be $n = \Pi P$, $n = 2\Pi P$ and $n = 4\Pi P$. Every factor of exponent n has Fermat's equation [1-7]. Using modular elliptic curves Wiles and Taylor prove FLT [9, 10]. This is not the proof that Fermat thought to have had.

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